THE LEFSCHETZ PROPERTY FOR FACE RINGS

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1. Introduction

The paper is devoted to proving the following theorem:

Theorem 1.1. *Fix an infinite field $\Bbbk$ of any characteristic. Consider a triangulated compact, closed, $\Bbbk$-orientable $\Bbbk$-homology manifold M of dimension $d-1$, and the associated graded commutative face ring $\Bbbk[M]$. Then the Gorensteinification of the generic Artinian reduction $\mathscr{A}^*(M)$ of $\Bbbk[M]$ has the weak Lefschetz property with respect to a generic $\ell \in \mathscr{A}^1(M)$, that is, for every $k \le d/2$, the map*

$$\mathscr{B}^k(M) \ \xrightarrow{\ \cdot\ell\ }\ \mathscr{B}^{k+1}(M)$$

is injective.

We call this the **generic weak Lefschetz property**. Here $\mathscr{B}^*$ is the Gorensteinification with respect to the unique socle element of degree d, that is, the minimal quotient of $\mathscr{A}^*$ that does not trivialize the degree d component of the ring. In the case that M is a $\Bbbk$-homology sphere, $\mathscr{A}^*(M) = \mathscr{B}^*(M)$.

Let us make two notes.

- It suffices to prove this for manifolds of even dimension $2k$, and for the map
$$\mathscr{B}^k(M) \ \xrightarrow{\ \cdot\ell\ }\ \mathscr{B}^{k+1}(M).$$
In this case, the map is an injection if and only if it is an isomorphism, as both spaces are of the same dimension, see Remark 4.9.
- One can extend this rather straightforwardly to the strong (or hard) Lefschetz property. We will do so at the end, as the weak Lefschetz property stated above is enough for almost all combinatorial applications. The hard Lefschetz property is, in the setting of Theorem 1.1, an isomorphism
$$\mathscr{B}^k(M) \ \xrightarrow{\ \cdot\ell^{d-2k}\ }\ \mathscr{B}^{d-k}(M).$$

Convention 1.2. Henceforth, a manifold is simply a triangulated $\Bbbk$-homology manifold over the chosen field $\Bbbk$, and a sphere is a $\Bbbk$-homology manifold with the $\Bbbk$-homology of a sphere. Furthermore, all simplicial complexes are finite, and hence all manifolds are compact. Homology

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This is an expository version of the author's original proof of the g-conjecture. It uses some simplifications and alternative constructions, sacrifices some generality of the algebraic theorems for simplicity, and works in arbitrary characteristic from the start. It still contains the derivations of other corollaries of the Lefschetz property, but otherwise cuts the exposition short. Supported by Horizon Europe ERC Grant number: 101045750 / Project acronym: HodgeGeoComb.

is defined over the given field, and only sometimes repeated for emphasis. Similarly, we will sometimes emphasize the nature of the manifold if some result *does not* extend to homology manifolds, etc. Orientability will be defined with respect to the chosen field. Similarly, Betti numbers, (co)homology, etc. are always stated with respect to the chosen field. All manifolds of positive dimension are connected.

1.1. **Applications to combinatorics.** We recall some of the consequences of Theorem 1.1 following [Adi18], though in a slightly greater generality from the start.

(1) *McMullen's g-conjecture, see* [McM71]*:* A direct application of Theorem 1.1, restricted to spheres, is the characterization of its face vector, following and resolving the g-conjecture of McMullen.

In particular, it proves that the f-vector, that is, the number of vertices, edges, two-dimensional faces, etc. of a given sphere Σ is also the f-vector of some simplicial polytope (or in other words, the set of f-vectors of spheres coincides with the set of f-vectors of simplicial polytopes). This is achieved by realizing, using the weak Lefschetz property, that the f-vector is but a nonnegative combination of the entries of the g-vector, that is, the numbers

$$g_i(\Sigma) \ = \ \dim \mathscr{A}^i(\Sigma)\big/\ell\mathscr{A}^{i-1}(\Sigma)$$

of $\mathscr{A}^*(\Sigma)$, see also [Sta80], where here and henceforth $\dim = \dim_{\Bbbk}$ unless otherwise made explicit.

The g_i are therefore the Hilbert vector of a commutative graded algebra generated in degree one (also known as M-vector), and it was proven by Billera and Lee [BL80] that, conversely, every M-vector can be realized as the g-vector of a simplicial polytope. Hence, face vectors of simplicial polytopes and face vectors of spheres coincide. We refer to Stanley's aforementioned work that discusses the g-conjecture, and its relation to the Lefschetz theorem in detail.

Going beyond spheres, the generic Lefschetz property provides necessary conditions on the possible face vectors of triangulations of a fixed manifold M as well.

(2) *Kühnel conjectures on Heawood inequalities:* In particular, the Lefschetz properties also apply to bound the complexity of triangulated (closed) manifolds, and resolve a conjecture of Kühnel [Küh95]: If M is a triangulated $(d-1)$-dimensional closed manifold on n vertices, then the generic Lefschetz property applied to each link implies that in terms of its reduced Betti numbers over $\Bbbk$, that is

$$\mathrm{b}_*(M) = \mathrm{b}_*(M;\Bbbk) = \dim \widetilde{H}_*(M;\Bbbk),$$

we have

$$\binom{d+1}{j}\mathrm{b}_{j-1}(M) \ \le \ \binom{n-d+j-2}{j} \quad \text{for } 1 \le j \le \frac{d}{2}.$$

This extends to manifolds with boundary, in a way. In that case, we have

$$\binom{d}{j}\mathrm{b}_{j-1}(M) \ + \ \binom{d}{j-1}\mathrm{b}_{d-j}(M) \ \le \ \binom{n-d+j-1}{j} \quad \text{for } 1 \le j \le \frac{d}{2},$$

though for non-orientable manifolds, this seemingly requires the Hard Lefschetz property. We provide a simple proof of both implications in Section 5. Compare also [Mur15] for earlier partial results (which are also conditional on the restrictive assumption that the manifold is locally polytopal).

(3) *Complexity measures for connectivity:* Given a sphere Σ of dimension $d-1$, and a nonnegative integer k less than d, one can measure the "complexity" of the triangulation in dimension $k-1$ in several ways. Consider for instance $\widetilde{H}_{k-1}(\Sigma_W)$ for W a subset of vertices $\Sigma^{(0)}$ of Σ and Σ_W the induced subcomplex [1] on vertex set W. For a numerical measure, one often uses either the 1-norm

$$|(\Sigma)|_{k-1,1,m} \coloneqq \sum_{\substack{W\subset \Sigma^{(0)}\\ |W|=m}} \mathrm{b}_{k-1}(\Sigma_W)$$

or the ∞-norm

$$|(\Sigma)|_{k-1,\infty} \coloneqq \max_{W\subset \Sigma^{(0)}} \mathrm{b}_{k-1}(\Sigma_W).$$

Following [MN03] and using the Lefschetz property, it follows that $|(\Sigma)|_{k-1,1,m}$ is minimized by the Billera-Lee polytopes among those with the same f-vector, see also Codenotti, Santos and Spreer [CSS18].

Moreover, it follows from [Adi17, Section 6] and the Lefschetz property that for $k \le d/2$, we have a bound

$$|(\Sigma)|_{k-1,\infty} \le\ g_k(\Sigma),$$

see also [ANS16]. Global combinatorial consequences of this are obtained if g_k vanishes, for instance the Generalized Lower Bound Theorem of Murai-Nevo [MN13] for general simplicial homology spheres.

In addition, there exists a subset $\mathscr{E} \subset \Sigma^{(0)}$,

$$|\mathscr{E}| \le\ ((k+1)g_{k+1} + (d+1-k)g_k)(\Sigma)$$

such that $\mathrm{b}_{k-1}(\Sigma_{W\cup\mathscr{E}}) = 0$ for all $W \subset \Sigma^{(0)}$. Hence, nontrivial homology classes in $H_{k-1}(\Sigma_W)$ are not only boundaries in Σ, but one can find them to be boundaries of chains with at most

$$((k+1)g_{k+1} + (d+1-k)g_k)(\Sigma)$$

additional vertices.

By adapting [Adi17, Section 6] to manifolds, one can show the two results mentioned previously still apply to measure the complexity of triangulated manifolds M, but now measuring not the induced homology of subcomplexes, but the dimension of the kernel of

$$\widetilde{H}_{k-1}(M_W) \ \longrightarrow\ \widetilde{H}_{k-1}(M).$$

(4) *Grünbaum-Kalai-Sarkaria conjecture, see* [Gr70]*:* A central application is to a conjecture of Grünbaum, Kalai and Sarkaria: The generic Lefschetz property for PL spheres implies that if Δ is a simplicial complex of dimension d that allows a PL embedding into $\mathbb{R}^{2d}$ (or, equivalently, S^{2d}) then

$$f_d(\Delta)\ \le\ (d+2)f_{d-1}(\Delta). \tag{1}$$

We will give a proof of this implication in Section 5.3, and in fact it motivates much of the inductive structure of the proof. This bound is essentially sharp, only differing from constructions in an additive error depending only on d, see also Remark 5.10. This does extend to spheres and manifolds, if we specify what we mean by PL embedding.

[1] That is, the subcomplex consisting of those simplices of Σ whose vertices are in a prescribed set.

The case $d = 1$ (when Δ is a simple graph) is a consequence of what is now called Euler's formula, postulated first by Descartes and Euler [Des21, Eul58], though the first formal proof by modern standards seems to be due to Legendre [Leg94].

However, already for $d = 2$, only a little improvement was known over the trivial bound of $f_2(\Delta) \le f_0(\Delta)f_1(\Delta)$: Dey [Dey93] proved that

$$f_d(\Delta) \;\le\; C f_0^{d+1-\frac{1}{3^d}}(\Delta),$$

a bound that stood essentially uncontested for 25 years. Parsa [Par18] recently improved this to

$$f_d(\Delta) \;\le\; C f_0^{d+1-\frac{1}{3^{d-1}}}(\Delta)$$

but the topological techniques employed are seemingly limited in potential.

Our Inequality (1) implies the stronger and optimal

$$f_d(\Delta) \;\le\; (d+2)\binom{f_0(\Delta)}{d}.$$

We should remark that these bounds are contingent on the map being PL, and will discuss the somewhat subtle issue of non-PL maps when we discuss the derivation from the Lefschetz property. Beyond the PL case, the best bounds seem to be due to Parsa [Par18].

The bound we obtain for PL maps has various consequences in discrete geometry (see also [Wag13, Section 3]). For instance, if Γ is a simplicial complex and $\Gamma \to \mathbb{R}^{2d}$ is a PL map that is locally injective, that is, it is injective on neighborhoods of vertices, then we have for the number of pairwise intersections of d-simplices, the d-th crossing number $\mathrm{cr}_d(\Gamma)$, the inequality

$$\mathrm{cr}_d(\Gamma) \;\ge\; \frac{f_d^{d+2}(\Gamma)}{(d+3)^{d+2} f_{d-1}^{d+1}(\Gamma)}$$

if $f_d(\Gamma) > (d+3)f_{d-1}(\Gamma)$, therefore obtaining a generalization of the influential crossing lemma of Ajtai, Chvátal, Leighton, Newborn and Szemerédi [ACNS82, Lei83]. We will discuss more details, and generalizations to manifolds, in Section 5.3.

2. Key ideas and overview

The idea for the proof is this: ***Section 4*** refines and reworks combinatorial commutative algebra to further the understanding of face rings in terms of simplicial cohomology, and ***Section 6*** uses this to show the Lefschetz property is equivalent to a nondegeneracy property of the Poincaré pairing at subspaces. We call this *the biased pairing property*.

We show in ***Section 7*** that it is easy to show the Lefschetz property for certain kinds of manifolds which have a nice decomposition. We in particular introduce weakly deconstructible spheres and balls and show the Lefschetz property for them.

The idea is now that the biased pairing property (which is equivalent to the Lefschetz property) does not have to be proven by proving the Lefschetz property on the original manifold directly, which may be hard to decompose, but can instead be proved on a different manifold. ***Section 8*** shows that indeed, we can replace the starting manifold with a simpler one, that has a nice decomposition (and one that is locally weakly deconstructible)[2]. ***Section 9*** just puts things together then, though with some further simplifications in the algebraic reduction.

3. Basic Notions

We set up some of the basic objects, and refer to [AY21, Lee96, Sta96] for a more comprehensive introduction. Especially the first reference follows our viewpoint closely. Starting with Section 4, we investigate the Poincaré pairing and socle. We fix an infinite field $\Bbbk$.

3.1. **Face rings.** If Δ is an abstract simplicial complex on groundset $[n] := \{1, \cdots, n\}$, let $I_\Delta := \langle \mathbf{x}^\mathbf{a} : \mathrm{supp}\,(\mathbf{a}) \notin \Delta \rangle$ denote the nonface ideal in $\Bbbk[\mathbf{x}]$, where $\Bbbk[\mathbf{x}] = \Bbbk[x_1, \cdots, x_n]$. As a general convention, all our simplicial complexes (unless they are relative) have the empty set as a face. This also means that homology and cohomology of simplicial complexes is naturally always reduced.

Let $\Bbbk^*[\Delta] := \Bbbk[\mathbf{x}]/I_\Delta$ denote the face ring of Δ. A collection of linear forms $\Theta = (\theta_1, \cdots, \theta_l)$ in the polynomial ring $\Bbbk[\mathbf{x}]$ is a **partial linear system of parameters** if

$$\dim_{\mathrm{Krull}} \Bbbk^*[\Delta] \big/ \Theta \Bbbk^*[\Delta] \ = \ \dim_{\mathrm{Krull}} \Bbbk^*[\Delta] - l,$$

for $\dim_{\mathrm{Krull}}$ the Krull dimension. If $l = \dim_{\mathrm{Krull}} \Bbbk^*[\Delta] = \dim \Delta + 1$, then Θ is simply a **linear system of parameters**, and the corresponding quotient $\Bbbk^*[\Delta]/\Theta\Bbbk^*[\Delta]$ is called an **Artinian reduction** of $\Bbbk^*[\Delta]$. Of course, there is a choice of Artinian reduction, and we say a statement is true for a **generic Artinian reduction** if it is true for a Zariski open set of Artinian reductions.

3.2. **The relative case.** A **relative simplicial complex** $\Psi = (\Delta, \Gamma)$ is a pair of simplicial complexes Δ, Γ with $\Gamma \subset \Delta$. The faces of $\Psi = (\Delta, \Gamma)$ are the faces of Δ not in Γ. If $\Psi = (\Delta, \Gamma)$ is a relative simplicial complex, then we can define the **relative face module**

$$\Bbbk^*[\Psi] \ := \ I_\Gamma \big/ I_\Delta$$

and its reduction $\Bbbk^*[\Psi]/\Theta\Bbbk^*[\Psi]$.

3.3. **Coordinates and properness.** Observe finally that Θ induces a map $\Delta^{(0)} \to \Bbbk^l$ by associating to the vertices of Δ the coordinates $\mathbf{V}_\Delta = (v_1, \cdots, v_n) \in \Bbbk^{l \times n}$, where $\mathbf{V}_\Delta \mathbf{x} = \Theta$. Hence, as is canonical when considering toric varieties, we identify a pair $(\Delta; \Theta)$ with a **geometric simplicial complex**, that is, a simplicial complex with a map of the vertices to $\Bbbk^l$.

Conversely, the canonical reduced face rings of a geometric simplicial complex are those induced by the linear system of parameters given by the geometric realization. The face ring of a **geometric simplicial complex** is considered with respect to its natural system of parameters induced by the coordinates.

A geometric simplicial complex in $\Bbbk^d$ is **proper** if the image of every k-face, with $k < d$, linearly spans a subspace of dimension $k + 1$. A sequence of linear forms is a (partial) linear system of parameters if the associated coordinatization is proper (this is an easy exercise, see for instance [Lee96,

[2] This is the main deviation from [Adi18], as the construction is both simpler and more general.

Section 4]). For the results of our paper, we always think of every simplicial complex as geometric and proper, that is, as coming with a proper coordinatization in a vector space over $\Bbbk$, and shall generally assume $(d-1)$-dimensional complexes to be realized in $\Bbbk^d$ unless otherwise stated, so that the associated collection of coordinatizing linear forms is a linear system of parameters.

All geometric simplicial complexes in this paper are given proper geometric realizations. For geometric simplicial complexes Ψ, we shall use the notation

$$\mathscr{A}^*(\Psi) \;:=\; {}^{\Bbbk^*[\Psi]}\big/_{\mathbf{V}_\Psi \mathbf{x} \Bbbk^*[\Psi]}.$$

The (ideal generated by) $\mathbf{V}_\Psi \mathbf{x}$ will therefore also be called the **coordinate ideal of parameters** for a geometric simplicial complex. Note that $\mathscr{A}^*(\Psi)$ is not necessarily Artinian if Ψ is of dimension $\geq d$.

3.4. **Conewise polynomials.** It is occasionally useful to remember an alternative model for face rings: the ring of conewise polynomials, see for instance [BBFK02, Bri97]. Consider a simplicial complex Δ as above, together with a proper realization over $\Bbbk$, that is, with vertex coordinates in some vector space V over $\Bbbk$. For every face σ, consider the algebra $\mathscr{P}(\sigma)$ of polynomial functions from the span of σ in V to $\Bbbk$, and denote by $\mathscr{P}(C)$ the algebra of elements in $\bigoplus_{\sigma\in\Delta} \mathscr{P}(\sigma)$ that commutes with restriction to subspaces:

$$(f_\sigma)_{|\mathrm{span}(\tau)} \;=\; f_\tau$$

It is an easy exercise to see that $\Bbbk[\Delta] \;\cong\; \mathscr{P}(C)$. This is achieved by sending an indeterminate x_v corresponding to the vertex v to the conewise linear function that vanishes on the conepoint and on every ray of C except for the one corresponding to v, where it takes value 1 on v itself. As both algebras are generated in degree one, this determines the desired isomorphism. Notice that in the realized case the linear system of parameters has a particularly nice image: it consists of the restrictions of global linear functions.

3.5. **The cone lemmas.** A crucial ingredient for the inductive structure is given by pullbacks to variables corresponding to vertices. Recall that the **star** and **link** of a face σ in Δ are the subcomplexes

$$\mathrm{st}_\sigma \Delta \;:=\; \{\tau : \exists \tau' \supset \tau,\ \sigma \subset \tau' \in \Delta\} \ \text{ and } \ \mathrm{lk}_\sigma \Delta \;:=\; \{\tau \backslash \sigma : \sigma \subset \tau \in \Delta\}.$$

For geometric simplicial complexes Δ, we shall think of the star of a face as a geometric subcomplex of Δ, and the link of a face σ as the geometric simplicial complex obtained by the projection with kernel spanned by σ. Let us denote the **deletion** of σ by $\Delta - \sigma$, the maximal subcomplex of Δ that does not contain σ. Let

$$\mathrm{st}^\circ_\sigma \Delta \;:=\; (\mathrm{st}_\sigma \Delta, \mathrm{st}_\sigma \Delta - \sigma).$$

We have the following two elementary lemmas.

Lemma 3.1 (Cone lemma I, see [Lee96, Thm. 7] or [TW00])**.** *For any vertex $v \in \Delta$, where Δ is a geometric simplicial complex in $\Bbbk^d$, and any integer k, we have an isomorphism*

$$\mathscr{A}^k(\mathrm{lk}_v \Delta) \;\cong\; \mathscr{A}^k(\mathrm{st}_v \Delta).$$

Lemma 3.2 (Cone lemma II, see [Adi17, Lem. 3.3])**.** *In the situation of the first cone lemma we have a natural isomorphism*

$$\cdot x_v : \mathscr{A}^k(\mathrm{st}_v\, \Delta) \longrightarrow \mathscr{A}^{k+1}(\mathrm{st}_v^\circ\, \Delta)$$

where x_v acts by multiplication.

4. Partition of unity and Poincaré duality

The approach to combinatorial commutative algebra introduced as the **partition complex** turns out to be quite fruitful. It in particular gives a new proof of Reisner's theorem (that was claimed by Stanley and Walker in [BGS82]). In particular, all basic facts used in this section are also reproven using the techniques introduced here, see also [AY21] for a more verbose version.

Recall that a simplicial complex Δ of dimension $d-1$ is **Cohen–Macaulay** (over $\Bbbk$) if the link of a $(k-1)$-dimensional face has nontrivial reduced $\Bbbk$-homology at most in dimension $d-1-k$. Following a theorem of Hochster [Sta96, Section II.4], this is equivalent to the following statement:

Theorem 4.1 (Reisner [Rei76])**.** *Consider the face ring $\Bbbk^*[\Delta]$ and any linear system of parameters Θ. Consider the associated Koszul complex $K^\bullet(\Theta)$.*

Then Δ is Cohen-Macaulay if and only if $\Bbbk^*[\Delta] \otimes K^\bullet(\Theta)$ carries homology only in final nontrivial component, where the homology is precisely ${}^{\Bbbk^*[\Delta]}\big/_{\Theta \Bbbk^*[\Delta]}$.

The latter property is the origin of the term Cohen-Macaulay, with our topological notion historically being introduced after Reisner's theorem.

We can now state:

Lemma 4.2 (Partition of unity)**.** *Consider a Cohen–Macaulay $(d-1)$-complex Δ in $\Bbbk^d$. Then, for every $k < d$, we have an injection*

$$\mathscr{A}^k(\Delta) \longhookrightarrow \bigoplus_{v \in \Delta^{(0)}} \mathscr{A}^k(\mathrm{st}_v\, \Delta) \tag{2}$$

Here $\cdot^{(i)}$ denotes the set of i-dimensional faces of a simplicial complex. We consider homology with $\Bbbk$-coefficients.

Proof. Let Θ denote the coordinate ideal of parameters for Δ. For the proof, we consider the Koszul complex $K^\bullet := K^\bullet(\Theta)$ and the chain complex $\widetilde{\mathscr{P}}^\bullet = \widetilde{\mathscr{P}}^\bullet(\Delta)$ defined as

$$0 \longrightarrow \Bbbk^*[\Delta] \longrightarrow \bigoplus_{v \in \Delta^{(0)}} \Bbbk^*[\mathrm{st}_v\, \Delta] \longrightarrow \cdots \longrightarrow \bigoplus_{F \in \Delta^{(d-1)}} \Bbbk^*[\mathrm{st}_F\, \Delta] \longrightarrow 0$$

with a natural choice of maps for the boundary operator: the choice of signs can be intuited easily once we realize that the degree zero component is naturally the Čech complex of Δ covered by the interiors of stars $\mathrm{st}_v \Delta$ [ES52].

With this, we obtain a chain complex as desired. In positive degree, we simply have an inclusion-exclusion of monomials, so the complex is exact. Let $\mathscr{P}^\bullet = \mathscr{P}^\bullet(\Delta)$ denote the complex

$$0 \longrightarrow \mathscr{A}^*(\Delta) \longrightarrow \bigoplus_{v \in \Delta^{(0)}} \mathscr{A}^*(\mathrm{st}_v\, \Delta) \longrightarrow \cdots \longrightarrow \bigoplus_{F \in \Delta^{(d-1)}} \mathscr{A}^*(\mathrm{st}_F\, \Delta) \longrightarrow 0.$$

Consider the double complex $K^\bullet \otimes \widetilde{\mathscr{P}}^\bullet$ and the associated total complex $\mathrm{Tot}^\bullet$, that is, the complex on summands $\mathrm{Tot}^\ell = \bigoplus_{i+j=\ell} K^i \otimes \widetilde{\mathscr{P}}^j$. We compute the homology of the total complex by examining the two filtrations of the double complex.

In the direction of the Koszul complex, the double complex is exact except for the final homology group (which is the Artinian reduction). Using this at all the proper vertices, we obtain that $H^n(\mathrm{Tot}^\bullet)$ is simply $H^n(\mathscr{P}^\bullet)$, or specifically for $n = d$, the kernel of the map (2).

Now, we need another observation: All spaces involved have another grading, the grading according to degree of polynomials. To not clutter notation, we shall specify the degree in words. For the following analysis, it is useful to remember the stratification along this degree: Consider the double complex in direction of $\widetilde{\mathscr{P}}^\bullet$. This is exact in positive degree, so that only the degree 0 component of the homology in degree 0 survives. Keeping in mind that the Koszul complex increases the degree, it follows that

$$H^k(\mathscr{P}^\bullet) \ \cong\ H^k(\mathrm{Tot}^\bullet) \ \cong\ (H^{d-1})^{\binom{d}{k}}(\Delta)(-k),$$

where $(+j)$ denotes a shift in degree by j. Hence, we see that $H^d(\mathscr{P}^\bullet) \cong (H^{d-1}(\Delta))(-d)$, which is nontrivial only in degree d. But the degree k component of $H^d(\mathscr{P}^\bullet)$ is isomorphic to the kernel of the Map (2), as desired. □

A corollary of this fact is Poincaré duality for simplicial homology spheres, a classical fact due to Hochster [Sta96].

Theorem 4.3. *Let Σ be a $(d-1)$-dimensional simplicial homology sphere in $\Bbbk^d$. Then $\mathscr{A}^*(\Sigma)$ is a Poincaré duality algebra.*

For this, notice:

Lemma 4.4. *In a triangulated sphere Σ of dimension $d-1$ in $\Bbbk^d$, we have an injection*

$$\mathscr{A}^*(\mathrm{st}^\circ_v \Sigma) \ \longhookrightarrow\ \mathscr{A}^*(\Sigma)$$

for any vertex v of Σ.

Proof. Consider the short exact sequence

$$0 \ \longrightarrow\ \Bbbk^*[\mathrm{st}^\circ_v \Sigma] \ \longrightarrow\ \Bbbk^*[\Sigma] \ \longrightarrow\ \Bbbk^*[\Sigma - v] \ \longrightarrow\ 0.$$

Computing the Artinian reduction, and noticing all three terms are Cohen-Macaulay gives the desired. □

Proof of Theorem 4.3. Partition of unity implies for all $k < d$ an injection

$$\mathscr{A}^k(\Sigma) \ \longhookrightarrow \bigoplus_{v\in\Sigma^{(0)}} \mathscr{A}^k(\mathrm{st}_v\, \Sigma) \ \cong \bigoplus_{v\in\Sigma^{(0)}} \mathscr{A}^{k+1}(\mathrm{st}^\circ_v\, \Sigma) \ \longhookrightarrow \bigoplus_{v\in\Sigma^{(0)}} \mathscr{A}^{k+1}(\Sigma)$$

where the middle map is induced by the second cone lemma for each vertex v of Σ. □

4.1. **Manifolds.** For triangulated spheres, we will be interested mostly in the intersection rings $\mathscr{A}^*$, as the above section shows that this results in Poincaré duality algebras already. For manifolds (over $\Bbbk$), we will have to go beyond that because of our desire to work strongly with the Poincaré pairing in the face ring. Trying to prove partition of unity for $\mathscr{A}^*(M)$ reveals an even more remarkable fact, as it reveals that the socles can be naturally identified using the spectral sequence coming from the two filtrations of the double complex in the proof of Lemma 4.2 above. Computing the second page of both filtrations (i.e., the total complex), and using the fact that stars of nontrivial faces are Cohen–Macaulay, we obtain:

Proposition 4.5. *For a triangulated $(d-1)$-dimensional manifold M in $\Bbbk^d$, we have an exact sequence*

$$0 \longrightarrow (H^{k-1})^{\binom{d}{k}}(M) \longrightarrow \mathscr{A}^k(M) \longrightarrow \bigoplus_{v\in M^{(0)}} \mathscr{A}^k(\mathrm{st}_v\, M). \tag{3}$$

More generally, the degree $(k-i)$-component of $H^{d-i}(\mathscr{P}^\bullet(M))$ is isomorphic to $(H^{k-1})^{\binom{d}{k-i}}(M)$.

Proof. The proof is virtually identical to the proof of Lemma 4.2: The fact that $\widetilde{\mathscr{P}}^\bullet$ is exact in positive degree is untouched, so the computation in that direction of the double complex is untouched. In the direction of the Koszul complex, notice that the $\Bbbk^*[\mathrm{st}_\sigma M] \times K^\bullet(\Theta)$ has homology concentrated in the last component for every non-empty face σ. Hence, we obtain $H^n(\mathrm{Tot}^\bullet) \;\cong\; H^n(\mathscr{P}^\bullet)$ for every $n \le d$. □

Let us note that Proposition 4.5 applies to Buchsbaum complexes. These are those simplicial complexes of dimension $d-1$ in which all links of vertices are Cohen-Macaulay of dimension $d-2$.

4.2. **The combinatorial Ishida complex and the map** $\hom$**.** It is useful to recall the combinatorial Ishida complex of Tay and Whiteley [Ish80], called the "rigidity cochain complex" in their work. For a vector space V over $\Bbbk$ of dimension d, let us denote the j-th exterior power by $\bigwedge^j V$. If σ is an i-tensor, then multiplication with it in the exterior algebra induces a natural map

$$\bigwedge^{j} V \to \bigwedge^{j+i} V.$$

Consider a geometric complex Δ in V, and a non-negative integer $k \le \dim V$. We then define the combinatorial Ishida (cochain) complex $I^\bullet[k]$ as

$$0 \longleftarrow \bigoplus_{\sigma\in\Delta^{(k-1)}} \textstyle\bigwedge^{d-k} V\Big/\ker\sigma \longleftarrow \bigoplus_{\tau\in\Delta^{(k-2)}} \bigwedge^{d-k} V\Big/\ker\tau \longleftarrow \cdots \longleftarrow 0$$

where we identify σ with its $|\sigma|$-tensor and the coboundary of σ is defined as a signed formal sum over the faces of Δ covering σ :

$$\delta[\sigma] \;=\; \sum_{\tau\prec\sigma} \mathrm{sign}[\tau,\sigma][\tau];$$

which is naturally linearly extended. Indeed, this is just a simplicial cochain complex with a bit more intricate weights. We first recall a classical result.

Lemma 4.6 ([Oda91])**.** *The top cohomology of this complex is isomorphic to $\mathscr{A}^k(\Delta)$.*

Note that $\bigwedge^{d-k} V\Big/\ker\sigma \cong \Bbbk$ for $(k-1)$-dimensional faces σ, and that $\bigwedge^{d-k} V\Big/\ker\tau \cong \Bbbk^{\binom{[d]}{k}}$ for $(k-2)$-dimensional faces τ. Therefore, if we consider the space $C^{k-1}(\Delta)\Big/\delta C^{k-2}(\Delta) \supset H^{k-1}(\Delta;\Bbbk)$, then, appropriately weighted, its $\binom{[d]}{k}$-th power naturally surjects onto the top cohomology of the Ishida complex, and therefore $\mathscr{A}^k(\Delta)$.

Hence, we obtain a map

$$\hom : H^{k-1}(\Delta)^{\binom{[d]}{k}} \longrightarrow \mathscr{A}^k(\Delta) \tag{4}$$

We now have two maps between face rings and simplicial cohomology, one provided by the Ishida complex and the other by Proposition 4.5 and the partition complex and it is natural to ask how they relate. The answer is simple: as the Čech complex inducing the map to homology in the partition complex is exactly the Čech complex associated to the simplicial cohomology given by the Ishida complex. Hence, the map hom is exactly the map inducing Isomorphism (3).

4.3. **Socles.** For a simplicial $(d-1)$-manifold M that is realized in $\mathbb{k}^d$,

$$\mathscr{Soc}(M) \subset \mathscr{A}^*(M)$$

shall denote the annihilator of the irrelevant ideal[3] in $\mathscr{A}^*(M)$, also known as the **socle** of $\mathscr{A}^*(M)$. If M is closed, orientable (over $\mathbb{k}$) and connected, then $\mathscr{Soc}^d$ is generated by one element, the fundamental class, and it coincides with the fundamental class for simplicial cohomology of M. We shall elaborate on the connection between cohomology and the socle in a few moments. We will denote by $\mathscr{Soc}^\circ(M)$ the restriction of $\mathscr{Soc}(M)$ to degrees $\leq d-1$. We call this the **open socle**.

Note that with this definition, we have immediately

Lemma 4.7. *In a triangulated orientable manifold M of dimension $d-1$ in $\mathbb{k}^d$, we have an injection*

$$\mathscr{A}^*(\mathrm{st}_v^\circ M) \hookrightarrow \mathscr{A}^*(M)$$

for any vertex v of M.

In particular, we have

$$\mathscr{Soc}(M) \cong H^d(\mathscr{P}^\bullet(M))$$

Proof. The second statement is an immediate corollary of the first. For the first, notice that it suffices by Theorem 4.3 to prove this for degree d: indeed, if α is an element in $\mathscr{A}^*(\mathrm{st}_v^\circ M)$ of degree less than d, there is a β in $\mathscr{A}^*(\mathrm{st}_v M)$ so that $\alpha \cdot \beta \neq 0$ in $\mathscr{A}^d(\mathrm{st}_v^\circ M)$ by Theorem 4.3.

Returning to degree d, we observe that by the isomorphism to simplicial cohomology we gathered from the Ishida complex, this is just the isomorphism $H^{d-1}(\mathrm{st}_v^\circ M) \to H^{d-1}(M)$. □

We have the following useful corollary:

Proposition 4.8. *For a closed orientable triangulated $(d-1)$-manifold M in $\mathbb{k}^d$,*

$$\mathscr{A}^*(M) \big/ \mathscr{Soc}^\circ(M)$$

is a Poincaré duality algebra. Equivalently, it satisfies partition of unity as stated above, that is, in degrees $k < d$, the map

$$\mathscr{A}^*(M) \big/ \mathscr{Soc}^\circ(M) \longrightarrow \bigoplus_{v \in M^{(0)}} \mathscr{A}^*(\mathrm{st}_v M)$$

[3] The ideal generated by the indeterminates.

is injective.

The point is now that Proposition 4.5 computes the open socle. As a general rule, we will use

$$\mathscr{B}^*(M) \ := \ \mathscr{A}^*(M)\Big/\ker^\circ\big[\mathscr{A}^*(M) \ \to \ \textstyle\bigoplus_{v\in M^{(0)}} \mathscr{A}^*(\mathrm{st}_v M)\big]$$

to denote the "partitionable" reduction of $\mathscr{A}^*(M)$ of a manifold M, where

$$\ker^\circ\left[\mathscr{A}^*(M) \ \to \bigoplus_{v\in M^{(0)}} \mathscr{A}^*(\mathrm{st}_v M)\right] \ = \ (H^d)^\circ(\mathscr{P}^\bullet(M))$$

is the restriction of $H^d(\mathscr{P}^\bullet(M))$ to elements of degree at most $\dim M$. This notion also naturally applies to pairs $(M, \partial M)$ of a manifold relative to its boundary.

Note that for closed orientable manifolds, Lemma 4.7 implies that $\mathscr{B}^*(M)$ is the quotient of $\mathscr{A}^*(M)$ by its open socle.

Remark 4.9. With this notion, it is also clear that it suffices to prove Theorem 1.1 for the middle isomorphism as announced earlier. Indeed, assuming we have proven for all $(d-1)$-spheres that the map

$$\mathscr{A}^j(\Sigma) \ \xrightarrow{\cdot\nu} \ \mathscr{A}^{j+q}(\Sigma)$$

is injective for generic Artinian reductions and generic elements ν of $\mathscr{A}^q$, then for all closed orientable manifolds of dimension larger than $d-1$, we also have

$$\mathscr{B}^j(M) \ \xrightarrow{\cdot\nu} \ \mathscr{B}^{j+q}(M).$$

This is due to the injection

$$\mathscr{B}^*(M) \ \hookrightarrow \bigoplus_{v\in M^{(0)}} \mathscr{B}^*(\mathrm{st}_v\, M)$$

for all degrees less or equal to the dimension of M.

We also obtain the following version of Lemma 4.7.

Lemma 4.10. *In any closed orientable manifold M of dimension $d-1$ in $\Bbbk^d$, we have an injection*

$$\mathscr{A}^*(\mathrm{st}_v^\circ M) \ \hookrightarrow \ \mathscr{B}^*(M)$$

for any vertex v of M.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (H^d)^\circ(\mathscr{P}^\bullet(M-v)) & \longrightarrow & \mathscr{A}^*(M-v) & \longrightarrow & \mathscr{B}^*(M-v) & \longrightarrow & 0 \\
 & & \wr\uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & (H^d)^\circ(\mathscr{P}^\bullet(M)) & \longrightarrow & \mathscr{A}^*(M) & \longrightarrow & \mathscr{B}^*(M) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & \mathscr{A}^*(\mathrm{st}_v^\circ M) & \longrightarrow & \mathscr{A}^*(\mathrm{st}_v^\circ M) & \longrightarrow & 0
\end{array}$$

The horizontal sequences are exact by definition. It follows from Proposition 4.5 that the left top vertical map is an isomorphism. Hence, the first two vertical sequences are exact, and by the snake lemma, so is then the third, as desired. $\square$

Remark 4.11. If M is closed and orientable, then $\mathscr{B}^*(M)$ is also the quotient of $\mathscr{A}^*(M)$ by the annihilator under Poincaré pairing to the fundamental class: The unique socle element $\mu \in \mathscr{A}^d(M)$ defines a map $\mu^* : \mathscr{A}^d(M) \longrightarrow \Bbbk$.

Spinning this further, if Δ is any geometric simplicial complex of dimension $d-1$ realized in $\Bbbk^d$, then $\mathscr{A}^d(\Delta) \cong H^{d-1}(\Delta)$. Homology classes μ in $H_{d-1}(\Delta)$ are in correspondence to maps

$$\mu^* : \mathscr{A}^d(\Delta) \cong H^{d-1}(\Delta) \longrightarrow \Bbbk.$$

We can then look at the quotient of $\mathscr{A}^*(\Delta)$ by the annihilator under Poincaré pairing to $\Bbbk$ via μ^*, and denote it by $\mathscr{B}^*(\mu)$. For closed, orientable manifolds $\mathscr{B}^*(\Delta) = \mathscr{B}^*(\mu)$, where the latter is its fundamental class. This viewpoint is taken in [APP23], where the Lefschetz property is proven for cycles over characteristic 2.

4.4. **Manifolds with boundary, and how to talk about Lefschetz in the boundary case.** The results above extend to manifolds with boundary, with a twist.

Corollary 4.12. *In any manifold M of dimension $d-1$ in $\Bbbk^d$ with boundary ∂M, let $M^\circ = (M, \partial M)$. We have an injection*

$$\mathscr{A}^*(\mathrm{st}_v^\circ M^\circ) \hookrightarrow \mathscr{B}^*(M^\circ)$$

for any vertex v of M.

We then obtain, by using Proposition 4.5:

Proposition 4.13. *Given an orientable connected manifold M of dimension $d-1$ (possibly with boundary), there is a perfect pairing*

$$\mathscr{B}^k(M) \times \mathscr{B}^{d-k}(M^\circ) \longrightarrow \mathscr{B}^d(M^\circ) \cong \Bbbk$$

where $M^\circ = (M, \partial M)$.

4.5. **$\Bbbk$-homeomorphisms.** A notion that will be useful for us going forward is the notion of a $\Bbbk$-homeomorphism.

Definition 4.14. Let $f : X \longrightarrow Y$ be a continuous map, and let $\Bbbk$ be a field. For each $q \geq 0$, let

$$\mathscr{L}_q^X(U) := H_q(X, X \setminus U; \Bbbk), \qquad \mathscr{L}_q^Y(V) := H_q(Y, Y \setminus V; \Bbbk)$$

denote the local homology sheaves on X and Y.

We say that f is a $\Bbbk$-*homeomorphism* if, for every $q \geq 0$, the natural morphism

$$f^{-1}\mathscr{L}_q^Y \longrightarrow \mathscr{L}_q^X$$

is an isomorphism of sheaves.

Of course, we shall mainly be interested in the case of simplicial (or piecewise linear) maps. This is useful in several ways. We note the following:

Proposition 4.15. *Consider $f : M \longrightarrow N$ a $\Bbbk$-homeomorphism of manifolds. Then f induces an isomorphism in cohomology.*

The following is also useful to illustrate the notion:

Proposition 4.16. *Any subdivision/refinement of a simplicial complex induces a $\Bbbk$-homeomorphism.*

Moreover, consider M a closed manifold, and let Z denote an acyclic subcomplex of M. Then, if the contraction of Z is a simplicial complex[4], *then the simplicial map induced by the contraction is a $\Bbbk$-homeomorphism.*

Of course, being $\Bbbk$-homeomorphic is not an equivalence relation, and not even symmetric. However, symmetrizing and passing to the transitive closure, we obtain an equivalence relation, which we call $\Bbbk$**-equivalence** if two complexes are connected by a sequence of $\Bbbk$-homeomorphisms.

5. Between combinatorics and algebra

We use this section to go over some applications of Theorem 1.1 and relatives, and also introduce some useful notation and more general perspective for the following chapters.

5.1. Numerical consequences of the Lefschetz theorem.

Let us consider the following first: Let Σ denote a simplicial sphere and let Δ denote a subcomplex of Σ. We define

$$\mathsf{k}_i(\Sigma, \Delta) \ \coloneqq \ \dim \mathscr{K}^i(\Sigma, \Delta),$$

where

$$\mathscr{K}^*(\Sigma, \Delta) \ \coloneqq \ \ker[\mathscr{A}^*(\Sigma) \to \mathscr{A}^*(\Delta)].$$

Note that these are just the squarefree monomial ideals generated by the monomials supported outside Δ.

Let us note the following consequence of Theorem 1.1.

Observation 5.1. *Consider Σ a $2k$-dimensional sphere in $\Bbbk^{2k+1}$. If $\mathscr{A}(\Sigma)$ has the weak Lefschetz property, then for every subcomplex Δ of Σ, we have*

$$\mathsf{k}_k(\Sigma, \Delta) \ \leq \mathsf{k}_{k+1}(\Sigma, \Delta). \tag{5}$$

and equivalently

$$\dim \mathscr{A}^k(\Delta) \ \geq \ \dim \mathscr{A}^{k+1}(\Delta)$$

This is a trivial consequence of the commutative square

$$\begin{array}{ccc} \mathscr{A}^k(\Sigma) & \xrightarrow{\cdot \ell} & \mathscr{A}^{k+1}(\Sigma) \\ \uparrow & & \uparrow \\ \mathscr{K}^k(\Sigma, \Delta) & \xrightarrow{\cdot \ell} & \mathscr{K}^{k+1}(\Sigma, \Delta) \end{array}$$

so that an isomorphism (and in particular injection) of the top map implies an injection of the bottom map.

[4] This can be ensured by suitably refining M, see [AP25, Section 1.2].

5.2. **The Geiringer-Laman Theorem.** Let us remind ourselves of the "original" generic Lefschetz theorem, known as Laman's rigidity criterion for planar graphs, because it gives an intuition for what happens here. Recall that a **Laman graph** is a graph G on n vertices and $2n-3$ edges such that every subgraph on q vertices has at most $2q-3$ edges.

Observation 5.2. *Assume G is a graph generically embedded into $\Bbbk^2$. Then it is a Laman graph if and only if for every subgraph H of G,*

$$\dim \mathscr{A}^1(H) \ \geq\ \dim \mathscr{A}^2(H),$$

or equivalently

$$\kappa_1(G,H) \ \leq \kappa_2(G,H).$$

Theorem 5.3 (Geiringer-Laman, [PG27, Lam70])**.** *Consider a Laman graph G. Then a generic realization of G in $\mathbb{R}^2$ with straight edges is infinitesimally rigid, that is, every infinitesimal deformation of G that preserves edge-lengths in the first order is a restriction of a global isometry.*

Let us restate the Geiringer-Laman Theorem, using [Lee96], in a more algebraic form:

Theorem 5.4. *For any generic embedding of a Laman graph G into $\Bbbk^2$, we have an isomorphism*

$$\mathscr{A}^1(G) \ \xrightarrow{\ \cdot\ell\ }\ \mathscr{A}^2(G),$$

where ℓ denotes any generic element of $\mathscr{A}^1(G)$. Conversely, any graph on v vertices and $2v-3$ edges admits such an embedding if and only if it is a Laman graph.

In the latter form, the Geiringer-Laman Theorem looks suspiciously like a Lefschetz theorem. Its proof follows a very nice idea; we call it the "decomposition principle", named after the decomposition theorem for maps between algebraic varieties [BBDG18].

Proof Sketch. Every Laman graph can be shown to be obtained from a single edge by the Henneberg moves. One connects two distinct vertices of a graph, the other connects three vertices, two of which are joined by an edge, and deletes the same edge. From the algebraic viewpoint, it is useful to place that new vertex in the second case in the interior of the edge to be deleted. These moves

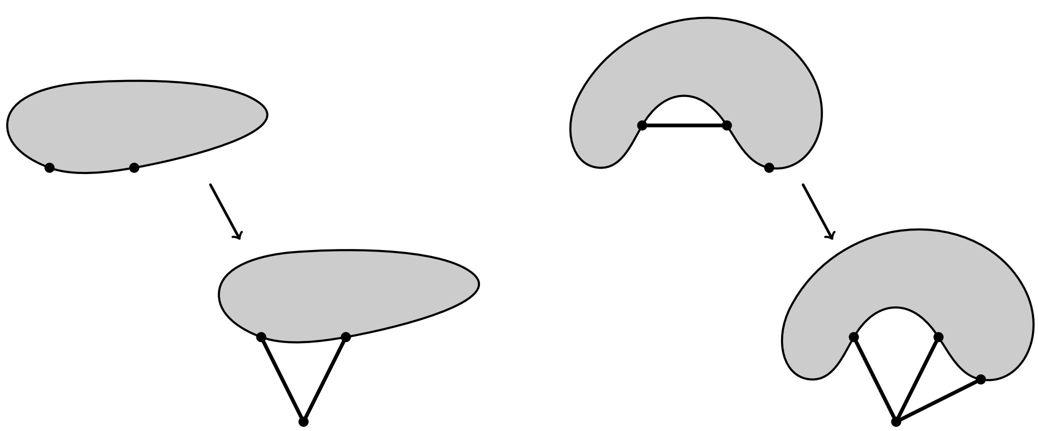

Figure 5.1. The two Henneberg moves.

preserve the "Lefschetz property", as seen by an appropriate form of the decomposition theorem: one constructs an embedding of algebras

$$\mathscr{A}^*(G) \hookrightarrow \mathscr{A}^*(G')$$

encoding the Henneberg move of G to G', gives an appropriate basis for the cokernel and proves the Lefschetz theorem on the decomposition, and in particular concludes rigidity for G' from rigidity for G. □

Hence, in the case of the Geiringer-Laman theorem, monotonicity of ideals (the numbers $\kappa_i(G, H)$) implies a Lefschetz type statement. The paper [Adi18], as well as [PP20, APP23] and the present paper, are devoted to first introducing, and then exploiting a trick to emulate this in higher dimension. Spoiler alert: We do not know yet whether a numerical monotonicity is enough. See also Question 10.3.

5.3. **The Grünbaum-Kalai-Sarkaria conjecture.** We now note a simple but important corollary of Theorem 1.1: The Grünbaum-Kalai-Sarkaria conjecture for sufficiently tame maps. We start with the case of spheres:

Corollary 5.5. *If Σ is a $2k$-dimensional sphere in $\Bbbk^{2k+1}$ that has the weak Lefschetz property, then, for every subcomplex Δ of Σ, we have Inequality (1), that is,*

$$(k+2) \cdot f_{k-1}(\Delta) \ \geq \ f_k(\Delta).$$

Proof. Observation 5.1 gives

$$\dim \mathscr{A}^k(\Delta) \ \geq \ \dim \mathscr{A}^{k+1}(\Delta).$$

Moreover, we have the immediate inequality

$$\dim \mathscr{A}^k(\Delta) \ \leq \ f_{k-1}(\Delta),$$

which follows by estimating the dimension of a vector space by the size of a generating set, in this case the $(k-1)$-faces and

$$\dim \mathscr{A}^{k+1}(\Delta) \ \geq \ f_k(\Delta) - (k+1) f_{k-1}(\Delta),$$

which follows by estimating the number of relations from above: By Oda's Lemma 4.6, there are $k+1$ relations for each $(k-1)$-face, so $(k+1)f_{k-1}(\Delta)$ is an upper bound for the number of nontrivial relations. □

We obtain the Grünbaum-Kalai-Sarkaria conjecture:

Corollary 5.6. *If Δ is a simplicial complex that PL embeds into $\mathbb{R}^{2d}$, then*

$$f_d(\Delta) \leq (d+2) f_{d-1}(\Delta)$$

This follows from Corollary 5.5 and the following folklore fact, see [AP25]:

Lemma 5.7. *Assume Δ is a simplicial complex of dimension d that allows a PL embedding into a manifold M. Then there is a PL triangulation of M that contains Δ as a subcomplex.*

However, not every topological embedding can be deformed to a PL one [DV09, Rus69]. That said, the results extend to appropriate notions of PL embeddings into homology spheres and homology manifolds, which illuminates the notion of "sufficiently tame".

Definition 5.8. We say a simplicial complex X $\mathbb{k}$**-embeds** into a simplicial complex Y if there is X' a $\mathbb{k}$-refinement of X, and Y' a complex $\mathbb{k}$-equivalent to Y, so that X' is a subcomplex of Y'. As natural, X' is a $\mathbb{k}$**-refinement** of X if there is a simplicial map $X' \twoheadrightarrow X$ whose fibers are $\mathbb{k}$-acyclic.

Examples of $\mathbb{k}$-embeddings include embeddings as subcomplexes, and PL embeddings (Lemma 5.7). We then obtain the following consequence of Theorem 1.1.

Corollary 5.9. *Consider a simplicial complex Δ which is $\mathbb{k}$-PL embeddable into a closed manifold of dimension $2d$. Then*

$$f_d(\Delta) \ \le\ (d+2)f_{d-1}(\Delta) + \binom{2d+1}{d}\mathrm{b}_d(M).$$

Proof. Note that it suffices to prove this in the orientable case. In a slight variation of Observation 5.1 and using Proposition 4.8, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{A}^d(M) \arrow[rr, "\cdot\ell"] \arrow[d, two heads] & & \mathscr{B}^{d+1}(M) \arrow[d] \\
\mathscr{A}^d(\Delta) \arrow[r, "\cdot\ell"] & \mathscr{A}^{d+1}(\Delta)\big/\hom H^d(M)^{\binom{2d+1}{d}} &
\end{tikzcd}$$

where the top map is a surjection by Theorem 1.1. $\square$

Remark 5.10. It is not hard to see that bound of Corollary 5.5 is tight up to an additive error that only depends on the dimension. One can determine the accurate bound by refining how we bounded the size of $\mathscr{A}^*(\Delta)$ in terms of its face numbers. For an alternative approach (that is conditional on the Lefschetz property we prove), see [Kal91, Kal01].

Remark 5.11 (Kühnel's Heawood type inequalities)**.** The inequality

$$\binom{2d+1}{d}\mathrm{b}_d(M) \ge \dim \mathscr{A}^{d+1}(\Delta) - \dim \mathscr{A}^d(\Delta)$$

from the proof of Corollary 5.9 has another consequence towards a conjecture of Kühnel [Küh94]: if a complete k-dimensional complex on n vertices embeds into a closed $2k$-manifold M, then

$$\binom{n-k-2}{k+1} \ \le\ \binom{2k+1}{k}\mathrm{b}_k(M),$$

thereby bounding the number of vertices such a complex can have. For topological embeddings where our methods don't apply, the best bounds were given by Paták and Tancer [PT24].

5.4. **Beyond closed manifolds.** We already touched upon generalizations to manifolds that are not orientable or closed in the previous section, but let us take a moment to actually formulate the main theorem, and their consequences for generalizing the previous section.

Theorem 5.12. *Fix an infinite field $\Bbbk$ of any characteristic. Consider a triangulated orientable $\Bbbk$-homology manifold M of dimension $d-1$, and the associated graded commutative face ring $\Bbbk[M]$. Then the Gorensteinification of a generic Artinian reduction of $\Bbbk[M]$ has the weak Lefschetz property with respect to a generic $\ell \in \mathscr{A}^1(M)$, that is, for every $k \geq \lfloor d/2 \rfloor$, the map*

$$\mathscr{B}^k(M) \xrightarrow{\cdot \ell} \mathscr{B}^{k+1}(M)$$

is surjective and if $d-1$ is even, then we have an isomorphism

$$\mathscr{B}^{(d-1)/2}(M, \partial M) \xrightarrow{\cdot \ell} \mathscr{B}^{(d+1)/2}(M).$$

Let us remark that similar hard Lefschetz properties exist. In the setting of this theorem, this is, for $k \leq d/2$, an isomorphism

$$\mathscr{B}^k(M, \partial M) \xrightarrow{\cdot \ell^{d-2k}} \mathscr{B}^{d-k}(M)$$

and in particular a surjection

$$\mathscr{B}^k(M) \xrightarrow{\cdot \ell^{d-2k}} \mathscr{B}^{d-k}(M).$$

We address and prove this property in Section 10.

We obtain the following generalizations of the previous section:

Corollary 5.13. *Consider a simplicial complex Δ which is $\Bbbk$-PL embeddable into an orientable manifold of dimension $2d$,*

$$f_d(\Delta) \ \leq \ (d+2) f_{d-1}(\Delta) + \binom{2d+1}{d} \mathrm{b}_d(M).$$

Moreover, if a complete k-dimensional complex on n vertices embeds into M, then

$$\binom{n-k-2}{k+1} \ \leq \ \binom{2k+1}{k+1} \mathrm{b}_k(M).$$

5.5. **Some conjectures of Kühnel.** Sometimes, the hard Lefschetz property is useful in combinatorial contexts as well. We now provide a simple way to conclude some conjectures of Kühnel [Küh95] from the hard Lefschetz property (Theorem 10.2) to illustrate the usefulness of the partition complex.

Theorem 5.14. *Consider M a triangulated manifold of dimension $d-1$ and on n vertices. Then*

$$\binom{d}{j} \mathrm{b}_{j-1}(M) \ + \ \binom{d}{j-1} \mathrm{b}_{d-j}(M) \ \leq \ \binom{n-d+j-1}{j} \quad \text{for } 1 \leq j \leq \frac{d}{2}.$$

If moreover M is closed, we have the stronger bound

$$\binom{d+1}{j} \mathrm{b}_{j-1}(M) \ \leq \ \binom{n-d+j-2}{j} \quad \text{for } 1 \leq j \leq \frac{d}{2},$$

and

$$\binom{d}{j} \mathrm{b}_{j-1}(M) \ \leq \ \binom{n-d+j-2}{j} \quad \text{for } 1 \leq j \leq d.$$

Proof. Let us first observe that by Macaulay's theorem [Mac27], it suffices entirely to prove that $\mathscr{A}^j(M)$ is of dimension at least

$$\binom{d}{j}\mathrm{b}_{j-1}(M) \;+\; \binom{d}{j-1}\mathrm{b}_{d-j}(M).$$

Consider now a generic element ℓ of $\mathscr{A}^1(M)$. Consider the diagram

$$\begin{tikzcd}
\mathscr{A}^j(M) \arrow[rr, "\cdot\ell^{d-2j+1}"] & & \mathscr{A}^{d-j+1}(M) \\
\bigoplus_{v\in M^{(0)}} \mathscr{A}^{j-1}(\mathrm{st}_v M) \arrow[u, two heads] \arrow[rr, two heads, "\cdot\ell^{d-2j+1}"] & & \bigoplus_{v\in M^{(0)}} \mathscr{A}^{d-j}(\mathrm{st}_v M) \arrow[u, two heads]
\end{tikzcd}$$

where the vertical maps are the cone lemmas, given by the composition

$$\mathscr{A}^{j-1}(\mathrm{st}_v M) \;\simeq\; \mathscr{A}^j(\mathrm{st}_v^\circ M) \;\longrightarrow\; \mathscr{A}^j(M)$$

where the last map is the inclusion of ideals. We notice that the bottom map is a surjection if $\mathscr{A}(\mathrm{st}_v M)$ has the Lefschetz property for balls. Hence, the top horizontal map is a surjection. By Proposition 4.5, the kernel of the top map is of dimension at least $\binom{d}{j}\mathrm{b}_{j-1}(M)$, and the image is of dimension at least $\binom{d}{j-1}\mathrm{b}_{d-j}(M)$.

Now, if M is even closed, we can go a step further: given the weak Lefschetz property in every link with respect to a generic ℓ, we can extend the Koszul complex by the generic Lefschetz element in the proof of Proposition 4.5 and obtain a short exact sequence

$$0 \;\longrightarrow\; (H^{k-1})^{\binom{d+1}{k}}(M) \;\longrightarrow \mathscr{A}^k(M)\big/\ell\mathscr{A}^{k-1}(M)$$

for $k \le d/2$. For arbitrary j, this additional partitioning property of coprimitive classes fails, so we only have the usual partitioning. These give the desired bound. $\square$

Remark 5.15. There are all kinds of other inequalities one can get from these kinds of arguments. For instance, we have, for a triangulated manifold M of dimension $d-1$ and on n vertices, that

$$\binom{d}{j}\mathrm{b}_{j-1}(M) \;+\; \binom{d}{j'-1}\mathrm{b}_{d-j'}(M) \;\le\; \binom{n-d+j-1}{j} \qquad \text{for } 1 \le j' \le j \le \frac{d}{2}.$$

5.6. **Partitioned quotients.** Before we continue, it is useful to extend the definition of the modules $\mathscr{K}$ to manifolds M. For subcomplexes Δ of M, we write

$$\kappa_i(M,\Delta) \;\coloneqq\; \dim \mathscr{K}^i(M,\Delta),$$

where

$$\mathscr{K}^*(M,\Delta) \;\coloneqq\; \ker[\mathscr{B}^*(M) \to \mathscr{B}^*(\Delta \subset M)]$$

and

$$\mathscr{B}^*(\Delta \subset M) \;\coloneqq\; \mathscr{A}^*(\Delta)\big/\ker\left[\mathscr{A}^*(M) \;\to\; \bigoplus_{v\in M^{(0)}} \mathscr{A}^*(\mathrm{st}_v M)\right].$$

It is a useful exercise to understand the dual of $\mathscr{K}^*(M,\Delta)$ under the Poincaré pairing: It is $\mathscr{B}^*(M)_{|\overline{\Delta}}$, the image of

$$\mathscr{B}^*(M) \;\longrightarrow \bigoplus_{\sigma\in M,\ \sigma\notin\Delta} \mathscr{A}^*(\mathrm{st}_\sigma M).$$

If M is closed, orientable and connected, this is precisely the Poincaré dual of $\mathscr{K}^*(M,\Delta)$, see Corollary 6.2. If M has boundary, the same applies with respect to the Poincaré duality for manifolds with boundary.

6. Biased Poincaré duality and the HGL relations: From Lefschetz to pairing properties and back

In this section, we introduce a crucial ingredient of the program by introducing a nondegeneracy property of the Poincaré pairing. This is the crucial ingredient to the proofs of the generic Lefschetz property, as they reduce the Lefschetz property to a pairing problem. It is the key to all known proofs of the generic Lefschetz property for spheres [Adi18, PP20, APP23], see Section 10.3.

6.1. **Poincaré pairings.** Recall: Let M be a triangulated (orientable, closed, connected) manifold of dimension $d-1$ realized in $\Bbbk^d$, and $\mathscr{B}^*(M)$ its face ring modulo Artinian reduction and open socle. Then we have a pairing

$$\mathscr{B}^k(M) \ \times\ \mathscr{B}^{d-k}(M) \ \longrightarrow\ \mathscr{B}^d(M)\ \cong\ \Bbbk.$$

6.2. **Biased Poincaré duality in ideals.** We say that M satisfies **biased Poincaré duality**, or has the **biased pairing property** in degree $k \le \frac{d}{2}$ if for all proper subcomplexes Δ of M, the pairing

$$\mathscr{K}^k(M,\Delta)\ \times\ \mathscr{K}^{d-k}(M,\Delta)\ \longrightarrow\ \mathscr{K}^d(M,\Delta)\ \cong\ \Bbbk \tag{6}$$

is nondegenerate on the first factor. Sometimes we will also say that a specific ideal $\mathscr{I}$ in a graded Poincaré duality algebra over $\Bbbk$ has the **biased pairing property**, if

$$\mathscr{I}^k\ \times\ \mathscr{I}^{d-k}\ \longrightarrow\ \mathscr{I}^d\ \cong\ \Bbbk \tag{7}$$

is nondegenerate in the first factor, where d is the degree of the fundamental class. An important special case is the biased pairing property at a subcomplex Δ, synonymous with the biased pairing property for the ideal $\mathscr{K}^*(M,\Delta)$.

We keep it separate as on its own it already has interesting consequences, even without having to consider a Lefschetz theorem. The following is but a restatement of the definition.

Proposition 6.1. *For an ideal $\mathscr{I}$ in $\mathscr{B}^*(M)$ the following are equivalent:*

(1) The map

$$\mathscr{I}\ \longrightarrow\ \mathscr{B}^*(M)\Big/\mathrm{ann}_{\mathscr{B}^*(M)}\mathscr{I}$$

is an injection in degree k.

(2) For every $x \in \mathscr{I}^k$, there exists a y in $\mathscr{I}^{d-k}$ such that $x \cdot y \neq 0$.

(3) $\mathscr{I}$ has the biased pairing property in degree k.

We obtain immediately an instrumental way to prove the biased pairing property for monomial ideals.

Corollary 6.2. *$\mathscr{K}^*(M,\Delta)$ has the biased pairing property in degree k if and only if*

$$\mathscr{K}^k(M,\Delta)\ \longrightarrow\ \mathscr{B}^k(M)_{|\overline{\Delta}}$$

is injective.

Proof. It is useful to understand what $\mathscr{B}^*(M)_{|\overline{\Delta}}$ is: following our definition, dualized, it is the image of

$$\mathscr{B}^*(M) \longrightarrow \bigoplus_{\sigma\in M,\ \sigma\notin\Delta} \mathscr{A}^*(\mathrm{st}_\sigma M).$$

Equivalently, it is the quotient of $\mathscr{B}^*(M)$ under the kernel of this map. Which is to say, it is the quotient of $\mathscr{B}^*(M)$ by the annihilator of $\mathscr{K}^k(M,\Delta)$ (as it is the ideal generated by the monomials not supported in Δ). Hence, the corollary follows from the previous proposition. $\square$

6.3. **Relation to the Grünbaum-Kalai-Sarkaria conjecture: Dimension reduction.** Assume that M is a (closed orientable) manifold of dimension $2k$ realized in $\mathbb{k}^{2k+1}$ so that the biased pairing property holds with respect to a subcomplex Δ in degree k. Then Inequality (5) follows, that is, we have

$$\mathsf{\kappa}_k(M,\Delta)\ \le\ \mathsf{\kappa}_{k+1}(M,\Delta).$$

In particular, the Grünbaum-Kalai-Sarkaria conjecture holds for Δ (with the appropriate correction term as in Remark 5.10).

6.4. **General properties of biased Poincaré duality.** To describe and prove the biased pairing property in general, we start with a first observation, the following persistence property.

Lemma 6.3. *Let M denote a closed orientable $(d-1)$-manifold in $\mathbb{k}^d$, and $k<\frac{d}{2}$. Then the biased pairing property holds for M in degree k if the biased pairing property holds for links of all vertices in M and in degree k.*

Proof. By partition of unity in M, any element of $\mathscr{K}^k(M,\Delta)$ pairs with some x_v. By Lemma 4.10, this pullback maps to $\mathscr{A}^k(\mathrm{lk}_v\, M)\cong x_v\mathscr{B}^k(M)$. Using the obvious fact that monomial ideals pull back to monomial ideals, we are reduced to proving Poincaré duality of $\mathrm{lk}_v\,\Delta$ in $\mathscr{A}^k(\mathrm{lk}_v\, M)$; the claim follows. $\square$

This allows us to prove the biased pairing property in degree k by restricting to the case of manifolds of dimension $d-1=2k-1$.

The biased pairing property is rather stable under combinatorial modifications. In the following lemma we focus on stability under subdivisions.

Lemma 6.4. *A PL homeomorphism $\varphi: M\to M'$ of closed orientable manifolds $M,\ M'$ in $\mathbb{k}^d$ that restricts to the identity on a common subcomplex Δ preserves the biased pairing property at Δ, that is, $\mathscr{K}^*(M,\Delta)$ has the biased pairing property (in degree k) if and only if $\mathscr{K}^*(M',\Delta)$ does (in degree k).*

Proof. By Lemma 6.3, we may assume that M is of dimension $2k-1$. The idea is to construct an isomorphism

$$\overline{\mathscr{K}}^*(M,\Delta)\ \cong\ \overline{\mathscr{K}}^*(M',\Delta)$$

where $\overline{\mathscr{K}}^*(M,\Delta)$ resp. $\overline{\mathscr{K}}^*(M',\Delta)$ are the orthogonal complements of $\mathscr{K}^*(M,\Delta)$ resp. $\mathscr{K}^*(M',\Delta)$ under the Poincaré pairing. If we do so, we are done, as proving the biased pairing property can be done by examining $\mathscr{K}^k(M,\Delta)$ or its orthogonal complement under the Poincaré pairing in degree k.

To this end, notice that we may define a PL homeomorphism more generally as any transformation obtained by stellar subdivisions and their inverses. Following Alexander's classical work (see [Lic99] for a survey), this subsumes classical PL equivalences[5]. Recall that a refinement of geometric simplicial complexes X to a complex X' induces a map

$$\mathscr{A}^*(X) \longrightarrow \mathscr{A}^*(X').$$

This is easily seen on the level of conewise polynomial functions, see Section 3.4. After all, a conewise polynomial on a simplicial cone complex is conewise polynomial on any refinement.

Now, a stellar subdivision of M at a face σ, denoted by $M{\uparrow}\sigma$, induces a pullback map

$$\mathscr{A}^*(M) \hookrightarrow \mathscr{A}^*(M{\uparrow}\sigma). \tag{8}$$

Its cokernel is a subspace of the ideal of x_{v_σ}: after all, it is generated by those functions that are conewise polynomial on $M{\uparrow}\sigma$, but not on M. But this means we can generate those functions by those that vanish on $M-\sigma$. Which are generated by x_{v_σ}.

Now, the embedding (8) above descends an embedding

$$\mathscr{B}^*(M) \hookrightarrow \mathscr{B}^*(M{\uparrow}\sigma).$$

This is simply because the kernels of $\mathscr{A}^*(M) \to \mathscr{B}^*(M)$ resp. $\mathscr{A}^*(M{\uparrow}\sigma) \to \mathscr{B}^*(M{\uparrow}\sigma)$ are identified by Proposition 4.5 as homology groups of M, which stay invariant in the subdivision. The cokernel is therefore[6], once again, lying in the ideal of x_{v_σ}. Hence, the cokernel is identical to the cokernel of

$$\mathscr{K}^*(M,\Delta) \hookrightarrow \mathscr{K}^*(M{\uparrow}\sigma,\Delta).$$

Hence, $\overline{\mathscr{K}}^*(M,\Delta) \cong \overline{\mathscr{K}}^*(M{\uparrow}\sigma,\Delta)$ stays invariant under the subdivision, which implies in particular that the non-degeneracy of the Poincaré pairing is preserved. $\square$

Remark 6.5. In fact, the argument above gives that

$$\mathscr{B}^*(M{\uparrow}\sigma) \cong \mathscr{B}^*(M) \oplus \bigoplus_{i=1}^{|\sigma|-1} x_{v_\sigma}^i \mathscr{A}^*(\mathrm{lk}_\sigma M).$$

Moreover, this decomposition is canonically orthogonal under the Poincaré pairing, see also [Pet16, Proposition 2.2]. For toric varieties, this is but a special case of the decomposition theorem (see [BBDG18]) and the proof in this generality is a straightforward adaptation.

Let us note that one can prove a stronger result, but we need a little more homological algebra.

Proposition 6.6. *A $\Bbbk$-homeomorphism $\varphi : M \to M'$ of closed orientable manifolds M, M' in $\Bbbk^d$ that restricts to the identity on a common subcomplex Δ preserves the biased pairing property at Δ, that is, $\mathscr{K}^*(M,\Delta)$ has the biased pairing property (in degree k) if and only if $\mathscr{K}^*(M',\Delta)$ does (in degree k).*

---

[5]Since we require geometric subdivisions rather than abstract topological ones, we invoke [AI15, Corollary 6] or [AP26] rather than just PL topology.

[6]It can be shown that the cokernel has not changed, though that is irrelevant for us.

Proof. We can assume again that both manifolds M resp. M' are of dimension $d-1=2k-1$ by Lemma 6.3, and that Δ is an induced subcomplex. For the latter, consider the biased derived subdivision of M resp. M' at Δ (see [AP25]), that is, we perform stellar subdivisions at all faces of M resp. M' that are not in Δ in reverse order of inclusion. We already saw that geometric stellar subdivisions do not affect the biased pairing property in the previous proof.

Now consider the mapping cylinder C of φ, and contract $\Delta \times [0,1]$ to Δ (this is possible as Δ is an induced subcomplex). Triangulate the result (without subdividing Δ), and call the result $\widetilde{\mathrm{C}}$. Give it a realization in $\mathbb{k}^d$ that extends the geometric realization of M resp. M' in the respective boundary components.

We have a chain complex

$$0 \longrightarrow \mathbb{k}^*[\widetilde{\mathrm{C}}] \longrightarrow \bigoplus_{v\in(\widetilde{\mathrm{C}},\Delta)^{(0)}} \mathbb{k}^*[\mathrm{st}_v\, \widetilde{\mathrm{C}}] \longrightarrow \cdots \longrightarrow \bigoplus_{F\in(\widetilde{\mathrm{C}},\Delta)^{(d-1)}} \mathbb{k}^*[\mathrm{st}_F\, \widetilde{\mathrm{C}}] \longrightarrow \bigoplus_{G\in(\widetilde{\mathrm{C}},\Delta)^{(d)}} \mathbb{k}^*[\mathrm{st}_G\, \widetilde{\mathrm{C}}] \longrightarrow 0$$

Consider also the chain complexes

$$0 \longrightarrow \mathbb{k}^*[M] \longrightarrow \bigoplus_{v\in(M,\Delta)^{(0)}} \mathbb{k}^*[\mathrm{st}_v\, M] \longrightarrow \cdots \longrightarrow \bigoplus_{F\in(M,\Delta)^{(d-1)}} \mathbb{k}^*[\mathrm{st}_F\, M] \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{k}^*[M'] \longrightarrow \bigoplus_{v\in(M',\Delta)^{(0)}} \mathbb{k}^*[\mathrm{st}_v\, M'] \longrightarrow \cdots \longrightarrow \bigoplus_{F\in(M',\Delta)^{(d-1)}} \mathbb{k}^*[\mathrm{st}_F\, M'] \longrightarrow 0.$$

The inclusion of M resp. M' into $\widetilde{\mathrm{C}}$ induce chain homotopies because φ is a $\mathbb{k}$-homeomorphism. Hence, the inclusions induce isomorphisms of the kernel of the map

$$\mathscr{A}^*(\widetilde{\mathrm{C}}) \longrightarrow \bigoplus_{v\in(\widetilde{\mathrm{C}},\Delta)^{(0)}} \mathscr{A}^*(\mathrm{st}_v\, \widetilde{\mathrm{C}})$$

to the kernels of

$$\mathscr{A}^*(M') \longrightarrow \bigoplus_{v\in(M',\Delta)^{(0)}} \mathscr{A}^*(\mathrm{st}_v\, M')$$

and

$$\mathscr{A}^*(M) \longrightarrow \bigoplus_{v\in(M,\Delta)^{(0)}} \mathscr{A}^*(\mathrm{st}_v\, M),$$

respectively. Now, notice that the biased pairing property in degree k for $\mathscr{K}^*(M,\Delta)$ resp. $\mathscr{K}^*(M',\Delta)$ is equivalent to the injectivity of

$$\mathscr{B}^*(M) \longrightarrow \mathscr{B}^*(\Delta\subset M) \oplus \bigoplus_{v\in(M,\Delta)^{(0)}} \mathscr{A}^*(\mathrm{st}_v\, M)$$

and

$$\mathscr{B}^*(M') \longrightarrow \mathscr{B}^*(\Delta\subset M') \oplus \bigoplus_{v\in(M',\Delta)^{(0)}} \mathscr{A}^*(\mathrm{st}_v\, M')$$

respectively, and the claim follows. □

6.5. **Characterization theorem for biased Poincaré duality: spherical envelopes.** We now explore characterizing the biased pairing property. This is our first result.

Proposition 6.7. *Assume E is a hypersurface sphere in a closed orientable manifold M of dimension $2k-1=d-1$. Then M in $\Bbbk^d$ has the biased pairing property in degree k and with respect to E if and only if*

$$\mathscr{A}^k(E) \ = \ 0.$$

Observation 6.8. *Note that $\mathscr{A}^k(E) \ = \ 0$ is a Lefschetz property: Project E to a hyperplane along a projection map π , and let ϑ denote the* ***height*** *of the vertices over that projection: M is realized in $\Bbbk^d \cong \Bbbk^{d-1}\times\Bbbk$. ϑ denotes the linear form encoding the coordinates in the latter factor of $\Bbbk^{d-1}\times\Bbbk$, so that together with the partial linear system encoded by the linear system generated by the coordinates in $\Bbbk^{d-1}$, we recover the linear system encoded by the linear system corresponding to the coordinates in $\Bbbk^d$.*

Then $\mathscr{A}^k(E) \ = \ 0$ if and only if ϑ induces the middle Lefschetz isomorphism on $\mathscr{A}^*(\pi E)$, that is

$$\mathscr{A}^{k-1}(\pi E) \ \xrightarrow{\ \cdot\vartheta\ } \ \mathscr{A}^k(\pi E)$$

is an isomorphism.

The proof is very simple, but illustrates a few important principles.

Proof. We have a short exact sequence

$$0 \ \longrightarrow \ \mathscr{K}^k(M,E) \ \longrightarrow \ \mathscr{B}^k(M) \ \longrightarrow \ \mathscr{A}^k(E) \ \longrightarrow \ 0$$

so we see that

$$\mathscr{B}^k(M) \ = \ \mathscr{K}^k(M,E) \ = \ \mathscr{B}^k(M)_{|\overline{E}}$$

in case the first space is trivial, so the claim simply follows by Poincaré duality.

For the converse, we consider the exact sequence

$$0 \ \longrightarrow \ \mathscr{K}^k(M,E) \ \longrightarrow \ \mathscr{B}^k(M) \ \longrightarrow \ \mathscr{B}^k(E\subset M) \ \longrightarrow \ 0.$$

Notice that because

$$H^{k-1}(M) \ \longrightarrow \ H^{k-1}(E)$$

is the trivial map, we have

$$\mathscr{B}^k(E\subset M) \ \cong \ \mathscr{A}^k(E)$$

by Proposition 4.5. Hence, a nontrivial $\alpha\in\mathscr{A}^k(M)$ that maps nontrivially to $\mathscr{A}^k(E)$ is in the cokernel of the map from the cohomology of E. Consider then the compactification $\widetilde{M}$ of $M\setminus E$ as a manifold with boundary, creating two boundary components isomorphic to E. As α is in the cokernel from the cohomology of E, it is in the cokernel under the cohomology $\widetilde{M}$ (using the inclusion of E into the latter). Hence, α, in either of the two copies of E, has nontrivial image in $\mathscr{B}^k(E\subset\widetilde{M})$ by Proposition 4.5. Under the simplicial map identifying the two boundary components of $\widetilde{M}$ again, the manifold $\widetilde{M}$ gets mapped to M, and $\mathscr{B}^k(M)_{|\overline{E}}$ gets mapped to $\mathscr{B}^k(\widetilde{M})$:

Notice that

$$\mathscr{A}^k(\mathrm{st}_\sigma\widetilde{M}) \ = \ 0$$

unless σ is an interior $(k-1)$-face of $\widetilde{M}$, so that we have a well-defined map

$$\bigoplus_{\substack{\sigma\in M,\\ \sigma\notin E}} \mathscr{A}^k(\mathrm{st}_\sigma M) \ \longrightarrow \ \bigoplus_{\sigma\in\widetilde{M}} \mathscr{A}^k(\mathrm{st}_\sigma\widetilde{M})$$

and in particular a map of the images of these two direct sums in $\mathscr{A}^k(\widetilde{M})$ resp. $\mathscr{A}^k(M)$, i.e. an embedding map

$$\mathscr{B}^k(M)_{|\overline{E}} \hookrightarrow \mathscr{B}^k(\widetilde{M}).$$

so that then α is nontrivial $\mathscr{B}^k(M)_{|\overline{E}}$.

Hence

$$\mathscr{K}^k(M, E) \longrightarrow \mathscr{B}_k(M)^{|\overline{E}}$$

has a nontrivial cokernel (as $\alpha = 0$ is nontrivial in the latter space). Because $\mathscr{B}^k(M)_{|\overline{E}}$ and $\mathscr{K}^k(M, E)$ are isomorphic as vector spaces (they are dual under the Poincaré pairing) the map also has a nontrivial cokernel, which implies that M does not satisfy the biased pairing property with respect to Δ by Corollary 6.2. □

Example 6.9. Consider the case $k = 1$, and Σ a sphere of dimension 1, realized in $\mathbb{R}^2$. If Δ is a 0-dimensional sphere in Σ, then $\mathscr{K}(\Sigma, \Delta)$ has the biased pairing property if and only if Δ does not lie on a line through the origin in $\mathbb{R}^2$.

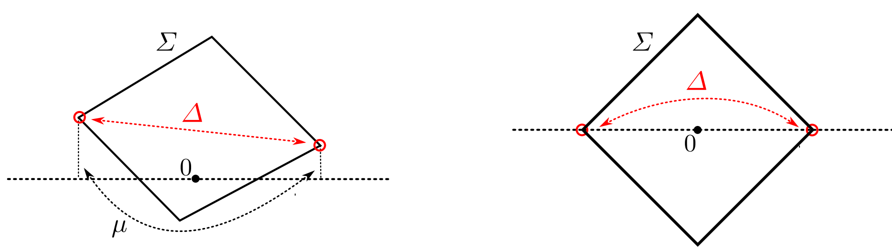

Figure 6.2. Biased Poincaré duality in a sphere Σ and with respect to a codimension one sphere Δ is related to the Lefschetz theorem on the latter.

On the other hand, this means that very nice and even smooth varieties do not satisfy the biased pairing property, for instance products of lower-dimensional varieties like the product of two projective lines $\mathbb{P}^1 \times \mathbb{P}^1$, shown in Figure 6.2 on the right.

6.6. **From spheres to complexes to manifolds.** Of course, not all subcomplexes Δ arise as codimension one spheres in a manifold. However, we will argue that proving the biased pairing property for an ideal $\mathscr{K}^*(M, \Delta)$ always reduces to considering hypersurfaces. As discussed in [Adi18], this can be reduced to the case when Δ is a special hypersurface with especially nice properties, and we will do so later, in Section 8. But for this illustrative section, which already presents the key ideas, we work with PL manifolds instead of the more complicated constructions later in this paper.

First, we observe that every complex of dimension $(k-1)$ in a combinatorial $(2k-1)$-manifold embeds into many hypersurfaces. This is based on a folklore geometric observation:

Proposition 6.10. *If Δ is a $(k-1)$-dimensional subcomplex in a $(2k-1)$-dimensional PL manifold M in $\Bbbk^{2k}$, then there exists a subdivision M' of M, and a $2(k-1)$-dimensional hypersurface S in M' containing Δ.*

A notion of importance in this context is the **simplicial neighborhood** $\mathrm{N}_\Gamma \Delta$ of an induced subcomplex Γ in a simplicial complex Δ, defined as

$$\mathrm{N}_\Gamma \Delta \ \coloneqq \bigcup_{w \in \Gamma^{(0)}} \mathrm{st}_w \, \Delta.$$

We call a simplicial neighborhood **regular** if it is regular in the sense of PL topology [**?**], that is, it is a PL mapping cylinder over the PL manifold given as the boundary of the simplicial neighborhood, mapped to Γ. We sometimes use the notion of regular neighborhood without referring to a specific Γ, and in this case it will simply be a regular simplicial neighborhood with respect to some Γ.

Proof. There exists a simplicial homeomorphism that embeds Δ into the boundary of its regular neighborhood in a sufficiently fine refinement of M by a folklore general position argument. This map can be assumed to be facewise linear on Δ by [AP25]. A subdivision of M that realizes the regular neighborhood therefore gives the desired subdivision. $\square$

Corollary 6.11. *In the situation of Proposition 6.10, there exists in a refinement M' of M a hypersurface with boundary $E_\Delta \supset \Delta$ so that*

$$\mathscr{A}^k(\Delta) \cong \mathscr{A}^k(E_\Delta).$$

We can assume that ∂E_Δ is an induced subcomplex of E_Δ.

We shall improve on this later, but it is a good illustration why reducing to hypersurfaces is enough. We need the following proposition, essentially due to Whiteley [Whi90], see also [BN10, Mur10]:

Proposition 6.12. *Consider a triangulated PL manifold S of dimension $2k-2$, realized in $\Bbbk^{2k-1}$. Consider an element $\ell \in \mathscr{A}^1(S)$, and an edge e of S. Assume that ℓ acts as a middle Lefschetz element on $\mathrm{lk}_e\, S$. Consider the stellar subdivision on the edge, and a generic linear combination ℓ' of ℓ with the variable of the new vertex v_e. Then*

$$\mathrm{coker}[\mathscr{A}^{k-1}(S) \ \xrightarrow{\cdot \ell}\ \mathscr{A}^k(S)] \ \hookrightarrow\ \mathrm{coker}[\mathscr{A}^{k-1}(S{\uparrow}e) \ \xrightarrow{\cdot \ell'}\ \mathscr{A}^k(S{\uparrow}e)].$$

Proof of Corollary 6.11. We can assume that the link of every edge not in E is combinatorially equivalent to a polytope by applying sufficiently many stellar subdivisions [AI15].

If E_Δ supports an element of degree k outside Δ then it must be supported in a $(k-1)$-face outside Δ. Removing that face may affect the manifold property, but we can refine E_Δ and M outside Δ so that the star of that face is a compact ball in the interior of E_Δ, in which case its removal preserves the manifold property.

To make sure this process terminates in finite time, we have to ensure that we do not introduce any new elements of degree k in the refinement; this is ensured by Proposition 6.12 and the Lefschetz theorem for spheres arising as boundaries of polytopes [Sta80]. $\square$

We shall need a stronger version of this corollary later, but we leave it here anyway because it serves to illustrate our strategy.

Let us make two quick remarks:

Remark 6.13. Instead of subdividing first to make links polytopal, we could of course simply use an induction and assume the Lefschetz theorem for $(2k-4)$-spheres already.

Remark 6.14. It is not hard to see that subdivisions at other faces similarly preserve the Lefschetz property (see [BN10]), but for simplicity we restrict to subdivisions of edges. In this case, which corresponds to the case of small resolution [Rei87], is especially nice in context of the Decomposition Theorem and Remark 6.5.

We call such a complex, that is, a supercomplex E of Δ such that $\mathscr{A}^k(\Delta) \cong \mathscr{A}^k(E)$ holds, an **envelope** in degree k for Δ. Passing from a subcomplex of M to its envelope (in degree k) does not affect the biased pairing property (in degree k) as in such case

$$\mathscr{K}^k(M, \Delta) \ \cong\ \mathscr{K}^k(M, E_\Delta).$$

Using the above, we can now replace any $(k-1)$-complex, such as the skeleton of a triangulated sphere, by a hypersurface. How does this help? First, we shall now see that a Lefschetz property for that hypersurface is equivalent to the desired biased pairing property.

6.7. **Characterization theorem for closed hypersurfaces.** We next want to deal with envelopes that are closed hypersurfaces. After all, we just argued that we can replace $(k-1)$-dimensional complexes in $(2k-1)$-dimensional manifolds with hypersurfaces (though, be careful, not generally closed). We nevertheless offer up this first intermediate case for rather didactic purposes; it is less technical than the case of hypersurfaces with boundary examined next.

We are almost in the same situation as in Proposition 4.5: M is a manifold of dimension $2k-1 = d-1$ in $\mathbb{k}^d$, but this time E is a closed hypersurface in it that is not necessarily a sphere. For simplicity, we shall assume such a hypersurface to be relatively $(k-2)$-acyclic (of course, with respect to $\mathbb{k}$), that is, the map $E \hookrightarrow M$ is injective in $\mathbb{k}$-homology up to dimension $k-2$. We will see that this is sufficiently general for our purposes. Indeed, following the proof of Proposition 6.10, this is a valid assumption we can make for PL manifolds: every Δ of dimension $k-1$ in M lies in a relatively $(k-2)$-acyclic closed hypersurface after subdivisions of M outside of Δ. We shall also assume that the map $E \hookrightarrow M$ is the trivial map in homology in dimension $d-2$, and shall treat the remaining case using the culminating methods of the next Section 6.8.

Note that then M is parted into a finite number of compact simplicial manifolds $(M_i)_{i\in I}$ such that the union of their boundaries is the hypersurface E. Note that if $k > 1$, then there are at most two of these manifolds. Note further that the biased pairing property with respect to E is equivalent to the biased pairing property with respect to

$$\overline{M}_j \ =\ \bigcup_{i\in I\setminus\{j\}} M_i$$

for all $j \in I$ if $|I| \geq 2$ as the ideals $\mathscr{K}(M, \overline{M}_j)$ are orthogonal to each other in the Poincaré pairing.

We may therefore restrict to examining $\overline{M}_1$ without loss of generality, and may assume that

$$E \ =\ \partial M_1 \ =\ \partial \overline{M}_1.$$

Theorem 6.15. *If $|I| \geq 2$, then a closed orientable $(d-1)$-manifold M in $\Bbbk^d$ has the biased pairing property with respect to $\overline{M}_1$ and degree $k = d/2$ as above if and only if the sequence*

$$(H^{k-1})^{\binom{d}{k}}(M) \longrightarrow (H^{k-1})^{\binom{d}{k}}(M_1) \longrightarrow \mathscr{B}^k(E \subset M) \longrightarrow 0$$

is exact.

The proof is a direct combination of Proposition 4.5 and Corollary 6.2.

Proof. The theorem follows at once from the commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \widetilde{\mathscr{K}}^k(M_1, E) \arrow[d] \arrow[dr, dashed] & & \\
(H^{k-1})^{\binom{d}{k}}(M) \arrow[r] & (H^{k-1})^{\binom{d}{k}}(M_1) \arrow[r] \arrow[dr, dashed, "\mathrm{hom}"] & \mathscr{B}^k(M_1 \subset M) \arrow[r] \arrow[d] & \mathscr{B}^k(M_1) \arrow[r] & 0 \\
 & & \mathscr{B}^k(E \subset M) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

where the top space $\widetilde{\mathscr{K}}^k(M_1, E)$ is defined by the vertical exact sequence. It injects into $\mathscr{K}^k(M, \overline{M}_1)$, which is the Poincaré dual to $\mathscr{B}^k(M_1)$ and therefore is isomorphic to it as vector space. Hence, the exact sequence of the assumption is equivalent to an isomorphism

$$\mathscr{B}^k(M_1) \cong \widetilde{\mathscr{K}}^k(M_1, E) \longleftarrow \mathscr{K}^k(M, \overline{M}_1),$$

where the spaces on the extreme left and extreme right are of the same dimension. Hence, the first isomorphism is therefore equivalent to an isomorphism between the extreme spaces, which is exactly the biased pairing property by Corollary 6.2. $\square$

6.8. **The doubling trick and the general characterization theorem.** To prove the biased pairing property for general subcomplexes, the above criterion will not be enough, unfortunately. In particular, we cannot always assume that E is a closed hypersurface. Instead, the criterion below will suffice in general (see also by Corollary 6.11). As for the setting: M is again a (closed, orientable) manifold of dimension $2k - 1 = d - 1$ in $\Bbbk^d$, but this time around we allow for hypersurfaces in it to have boundary.

Consider for this purpose the complement $\widetilde{M}$ of E in M, where E is a relatively $(k-2)$-acyclic orientable hypersurface in M whose boundary ∂E is an induced subcomplex of E, and the double $\mathrm{D}E$ of E. Notice that the proof of Corollary 6.11 guarantees the existence of such an envelope in degree k for any $(k-1)$-dimensional subcomplex of a combinatorial M (potentially having to subdivide M outside of that complex).

There is a **folding map**

$$\tau : \mathrm{D}E \longrightarrow E$$

for doubles that will be tremendously useful, which identifies the two copies of E that we call the **charts** of $\mathrm{D}E$.

The open manifold $\widetilde{M}$ can be compactified canonically to a compact manifold $\widetilde{\mathrm{D}}M$ with boundary $\mathrm{D}E$ (since E was orientable), and the folding map extends to a map

$$\tau : \widetilde{\mathrm{D}}M \longrightarrow M.$$

Let us also introduce the following useful concept: Let π denote the general position projection to a hyperplane H, and ϑ the height over that projection, so that

$$\mathscr{A}^k(X) = {}^{\mathscr{A}^k(\pi X)}\Big/_{\vartheta \mathscr{A}^{k-1}(\pi X)}$$

for a complex X in $\Bbbk^d$. Let us first observe the following helpful lemma that translates the problem to the case of doubles. It, in itself, is using Lefschetz properties.

Lemma 6.16. *Consider π a general position projection to a hyperplane, and let ϑ denote the height over that hyperplane. The following are equivalent:*

(1) The map

$$\mathscr{B}^{k-1}(\pi E, \partial \pi E) \xrightarrow{\cdot\vartheta} \mathscr{B}^k(\pi E \subset \pi M),$$

which factors as

$$\begin{tikzcd} \mathscr{B}^{k-1}(\pi E, \partial \pi E) \arrow[r] \arrow[d] & \mathscr{B}^k(\pi E \subset \pi M) \\ \mathscr{B}^{k-1}(\pi E \subset \pi M) \arrow[ur, "\cdot\vartheta"] & \end{tikzcd}$$

is an injection.

(2) The isomorphism

$$\mathscr{A}^k(M, E) \xrightarrow{\sim} \mathscr{A}^k(\widetilde{\mathrm{D}}M, \mathrm{D}E)$$

descends to an isomorphism of the image $\mathscr{K}^k(M, E)$ of

$$\mathscr{A}^k(M, E) \longrightarrow \mathscr{B}^k(M).$$

to the image $\mathscr{K}^k(\widetilde{\mathrm{D}}M, \mathrm{D}E)$ of

$$\mathscr{A}^k(\widetilde{\mathrm{D}}M, \mathrm{D}E) \longrightarrow \mathscr{B}^k(\widetilde{\mathrm{D}}M \subset M).$$

Here we read $\mathscr{B}^k(X \subset M)$ for $X \subset \widetilde{\mathrm{D}}M$ as the cokernel of the composition

$$\begin{tikzcd} (H^{k-1})^{\binom{d}{k}}(M) \arrow[r] \arrow[d, "\tau"] & \mathscr{A}^k(X) \\ (H^{k-1})^{\binom{d}{k}}(\widetilde{\mathrm{D}}M) \arrow[ur] & \end{tikzcd}$$

Moreover, let us note that for a manifold X with boundary ∂X, we define $\mathscr{B}(X, \partial X)$ naturally using the definition of stars in relative complexes:

$$\mathrm{st}_v(X, \partial X) \ = \ (\mathrm{st}_v X, \mathrm{st}_v \partial X).$$

Let us briefly take a moment to explain why we require Lemma 6.16, for all its technicality: We wish to prove the existence of an injection

$$\mathscr{K}^k(M, E) \ \hookrightarrow \ \mathscr{B}^k(M)_{|\overline{E}}.$$

Unfortunately, the first space, a partitioned quotient of $\mathscr{B}^k(M)$, lacks a nice description. Therefore, we look at doubles. In a déjà-vu from Proposition 6.7, notice that

$$\mathscr{A}^k(\mathrm{st}_\sigma \widetilde{\mathrm{D}}M) \ = \ 0$$

unless σ is an interior $(k-1)$-face of $\widetilde{\mathrm{D}}M$, so that we have a well-defined map

$$\bigoplus_{\substack{\sigma \in M^{(k-1)}, \\ \sigma \notin E}} \mathscr{A}^k(\mathrm{st}_\sigma M) \ \longrightarrow \ \bigoplus_{\sigma \in \widetilde{\mathrm{D}}M^{(k-1)}} \mathscr{A}^k(\mathrm{st}_\sigma \widetilde{\mathrm{D}}M)$$

and in particular a map

$$\mathscr{B}^k(M)_{|\overline{E}} \ \longrightarrow \ \mathscr{B}^k(\widetilde{\mathrm{D}}M).$$

We can now see how the doubling construction helps: Consider the commutative diagram

$$\begin{tikzcd}
\mathscr{K}^k(M, E) \arrow[r] \arrow[d] & \mathscr{K}^k(\widetilde{\mathrm{D}}M, \mathrm{D}E) \arrow[d] \\
\mathscr{B}^k(M)_{|\overline{E}} \arrow[r] \arrow[d, hook] & \mathscr{B}^k(\widetilde{\mathrm{D}}M) \arrow[d, hook] \\
\bigoplus_{\substack{\sigma \in M^{(k-1)}, \\ \sigma \notin E}} \mathscr{A}^k(\mathrm{st}_\sigma M) \arrow[r] & \bigoplus_{\sigma \in \widetilde{\mathrm{D}}M^{(k-1)}} \mathscr{A}^k(\mathrm{st}_\sigma \widetilde{\mathrm{D}}M).
\end{tikzcd}$$

The previous lemma ensures that the top horizontal map is an injection. Hence, proving an injection of the top-left vertical map is reduced to proving an injection on the top-right vertical map.

Proof of Lemma 6.16. We prove that (1) implies (2); this is, for our purposes, the critical direction. The converse is obtained by reversing the logic of the steps below.

The map described in (2) is clearly a surjection as it is the specialization of a surjective map. But the condition of (1) ensures that the kernels A of

$$\mathscr{B}^{k-1}(\pi E \subset \pi M) \ \xrightarrow{\cdot \vartheta} \ \mathscr{B}^k(\pi E \subset \pi M)$$

and B of

$$\mathscr{B}^{k-1}(\pi \mathrm{D}E \subset \pi M) \ \xrightarrow{\cdot \vartheta} \ \mathscr{B}^k(\pi \mathrm{D}E \subset \pi M)$$

inject into $\mathscr{A}^{k-1}(\partial \pi E)$, which in particular implies a surjection (and in fact an isomorphism)

$$A \ \twoheadrightarrow \ B.$$

On the other hand, we have by relative acyclicity the following commutative diagram for degrees at most k:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{A}^*(\pi\widetilde{\mathrm{D}}M, \pi\mathrm{D}E) \arrow[r] & \mathscr{B}^*(\pi\widetilde{\mathrm{D}}M \subset \pi M) \arrow[r] & \mathscr{B}^*(\pi\mathrm{D}E \subset \pi M) \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{A}^*(\pi M, \pi E) \arrow[r] \arrow[u] & \mathscr{B}^*(\pi M) \arrow[r] \arrow[u] & \mathscr{B}^*(\pi E \subset \pi M) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

which after reduction by ϑ implies a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] & \mathscr{A}^k(\widetilde{\mathrm{D}}M, \mathrm{D}E) \arrow[r] & \mathscr{B}^k(\widetilde{\mathrm{D}}M \subset M) \\
0 \arrow[r] & A \arrow[r] \arrow[u, two heads] & \mathscr{A}^k(M, E) \arrow[r] \arrow[u] & \mathscr{B}^k(M) \arrow[u]
\end{tikzcd}$$

which in turn implies that the desired map

$$\mathscr{A}^k(M, E) \longrightarrow \mathscr{A}^k(\widetilde{\mathrm{D}}M, \mathrm{D}E)$$

is also a surjection when restricted to images in $\mathscr{B}^k(M)$ and $\mathscr{B}^k(\widetilde{\mathrm{D}}M \subset M)$, respectively. $\square$

This takes us rather far, as we now have to prove that $\mathscr{B}(\widetilde{\mathrm{D}}M)$ is the image of $\mathscr{K}(\widetilde{\mathrm{D}}M, \mathrm{D}E)$ in degree k. We combine this with the arguments for Theorem 6.15 to obtain:

Theorem 6.17. *In the situation of the previous Lemma 6.16,*

$$(H^{k-1})^{\binom{d}{k}}(M) \longrightarrow (H^{k-1})^{\binom{d}{k}}(\widetilde{\mathrm{D}}M) \longrightarrow \mathscr{B}^k(\mathrm{D}E \subset M)$$

is exact if and only if

$$\mathscr{K}^k(\widetilde{\mathrm{D}}M, \mathrm{D}E) \longrightarrow \mathscr{B}^k(\widetilde{\mathrm{D}}M)$$

is injective.

Proof. Consider as in the proof of Theorem 6.15 the commutative diagram

$$\begin{tikzcd}
& & 0 \arrow[d] & & \\
& & \mathscr{K}^k(\widetilde{\mathrm{D}}M, \mathrm{D}E) \arrow[d] \arrow[dr, dashed] & & \\
(H^{k-1})^{\binom{d}{k}}(M) \arrow[r] & (H^{k-1})^{\binom{d}{k}}(\widetilde{\mathrm{D}}M) \arrow[r] \arrow[dr, dashed, "\mathrm{hom}"] & \mathscr{B}^k(\widetilde{\mathrm{D}}M \subset M) \arrow[r] \arrow[d] & \mathscr{B}^k(\widetilde{\mathrm{D}}M) \arrow[r] & 0 \\
& & \mathscr{B}^k(\mathrm{D}E \subset M) \arrow[d] & & \\
& & 0 & &
\end{tikzcd}$$

The vertical sequence is exact by definition, and the horizontal sequence is exact by Proposition 4.5. The claim follows at once by considering the diagonal maps. $\square$

6.9. **The HGL relations.** Finally, the biased pairing property allows us to formulate a Lefschetz property at ideals. We say that M a closed orientable manifold of dimension $d-1$ in $\mathbb{k}^d$ has the **HGL relations** (short for Hall, Geiringer and Laman) in degree $k \leq \frac{d}{2}$ and with respect to an ideal $\mathscr{I}^* \subset \mathscr{B}^*(M)$ if there exists an ℓ in $\mathscr{B}^1(M)$, the pairing

$$\begin{array}{ccccc} \mathscr{I}^k & \times & \mathscr{I}^k & \longrightarrow & \mathscr{I}^d \\ a & & b & \longmapsto & ab\ell^{d-2k} \end{array} \tag{9}$$

is nondegenerate. We will say M has the **HGL relations** if it does so at all squarefree monomial ideals, that is, for all $\mathscr{K}^*(M,\Delta)$, where Δ is any subcomplex of M.

$$\overline{\mathscr{K}}^*(M,\Delta) \ \coloneqq\ \ker\left[\mathscr{B}^*(M) \ \longrightarrow \ \bigoplus_{\substack{\sigma\in M\\ \sigma\notin\Delta}} \mathscr{A}^*(\mathrm{st}_\sigma M)\right].$$

Biased Poincaré duality is evidently a weaker form of the HGL relations, with equivalence in the case $k = d-k$.

7. Perturbations and Lefschetz elements via biased Poincaré duality

To prove the biased pairing property in higher degrees, we need to understand the Lefschetz theorem for face rings of manifolds; this in itself is not a hard task, but requires some care as to how we actually construct Lefschetz elements. The idea is provided in the following lemma, that we shall explore throughout the section.

7.1. **The Kronecker perturbation lemma.** The following lemma is probably classical, but was at least known to Kronecker [Kro90], see also [Rin13] for a statement in form of representation theory of the Kronecker quiver.

Lemma 7.1. *Consider two linear maps*

$$\alpha,\beta : \mathscr{X} \longrightarrow \mathscr{Y}$$

of two vector spaces $\mathscr{X}$ and $\mathscr{Y}$ over $\mathbb{k}$.

(1) Assume that β has image transversal to the image of α, that is,

$$\beta(\ker\alpha)\ \cap\ \mathrm{im}\alpha\ =\ 0\ \subset\ \mathscr{Y}.$$

Then a generic linear combination α“+”β of α and β has kernel

$$\ker(\alpha \text{ “+” } \beta)\ =\ \ker\alpha\ \cap\ \ker\beta.$$

(2) Similarly, if

$$\beta^\diamondsuit(\ker(\alpha^\diamondsuit))\ \cap\ \mathrm{im}(\alpha^\diamondsuit)\ =\ 0\ \subset\ \mathscr{X}^\diamondsuit$$

or equivalently

$$\beta^{-1}(\mathrm{im}\alpha)\ +\ \ker\alpha\ =\ \mathscr{X},$$

then

$$\mathrm{im}(\alpha \text{ “+” } \beta)\ =\ \mathrm{im}\alpha + \mathrm{im}\beta.$$

Here we use $(\cdot)^{\diamondsuit}$ to denote dual maps and vector spaces in the basic linear-algebra sense of the notion.

We will now discuss how this lemma is used in our context, but let us remark that many earlier results we referenced, such as those of Murai and Whiteley (Proposition 6.12), can be thought of as employing a very special case of Kronecker's lemma.

7.2. **The transversal prime property.** The strategy for the Lefschetz property, proven via the biased pairing property, is to prove the following property:

Let M denote a $(d-1)$-manifold (possibly with boundary) in $\Bbbk^d$, and $k \le \frac{d}{2}$. Let W denote a subset of the vertices of M. Then M has the **transversal prime property** in degree k and with respect to W if

$$\begin{aligned}
&\ker\left[\left(\text{``}\sum_{w\in W}\text{''}x_w\right)^{d-2k}: \mathscr{B}^k(M) \longrightarrow \mathscr{B}^{d-k}(M,\partial M)\right]\\
&= \operatorname{ann}_{\mathscr{B}^k(M)}\left\langle x_w | w\in W\subset M^{(0)}\right\rangle\\
&= \bigcap_{w\in W}\ker\left[x_w : \mathscr{B}^k(M) \longrightarrow \mathscr{B}_{k+1}(M)\right]
\end{aligned}$$

We remind the reader that

$$\text{``}\sum_{w\in W}\text{''}x_w$$

stands for a generic linear combination of indeterminates x_w with $w\in W$.

Note that if the transversal prime property holds for $W = M^{(0)}$, then we conclude the hard Lefschetz theorem for M. The general strategy for us is to prove the transversal prime property by linearly ordering the vertices in any way, and proceeding by induction on the size of initial segments.

7.3. **The perturbation approach.** Let us now explore how to apply Kronecker's lemma. Consider M a manifold with boundary of dimension $(d-1)$ realized in $\Bbbk^d$, and assume that we proved the transversal prime property for a set of vertices W' of M for the map between degree $d-k$ and degree k. For simplicity of notation, we will assume that $d = 2k+1$, the other Lefschetz maps will be reduced to that case. Consider an additional vertex of M not in W'.

When we want to prove the transversal prime property for $W = W' \cup \{w\}$, we want to use Lemma 7.1, applied to the maps

$$\alpha = \text{``}\sum_{v\in W'}\text{''}x_v \quad \text{and} \quad \beta = x_w$$

and to the spaces

$$\mathscr{X} = \mathscr{B}^k(M) \quad \text{and } \mathscr{Y} = \mathscr{B}^{k+1}(M,\partial M).$$

Now, note that if we have already proven the transversal prime property with respect to W', then kernel and image of α have a particularly nice description:

$$\ker\alpha = \bigcap_{v\in W'}\ker\left[x_v : \mathscr{B}^k(M,\partial M) \longrightarrow \mathscr{B}^{k+1}(M)\right]$$

and

$$\mathrm{im}\alpha \ = \ \mathrm{im}\left[\bigoplus_{v\in W'} \mathscr{A}^k(\mathrm{st}_v M) \longleftarrow \mathscr{B}^k(M)\right].$$

We have the following central theorem:

Theorem 7.2. *In the situation above, the following are equivalent.*

(1) The assumptions of Lemma 7.1(1) hold, that is,

$$\beta(\ker\alpha) \ \cap \ \mathrm{im}\alpha \ = \ 0 \ \subset \ \mathscr{Y}.$$

(2) The assumptions of Lemma 7.1(2) hold, that is,

$$\beta^{-1}(\mathrm{im}\alpha) \ + \ \ker\alpha \ = \ \mathscr{X}.$$

(3) The pullback of $\ker\alpha$ *to*

$$\mathscr{A}^k(\mathrm{st}_w M) \ \cong \ \mathscr{A}^k(\mathrm{lk}_w\, M)$$

along x_w *has the biased pairing property in the* $(d-2)$*-sphere* $\mathrm{lk}_w\, M$.

Proof. We have $(1) \Leftrightarrow (2)$ by Poincaré duality. For $(3) \Leftrightarrow (1)$, observe that $\ker\alpha$ and $\mathrm{im}\alpha$ are orthogonal complements in $\mathscr{B}^*(M)$. Hence

$$\mathscr{I}_W \ := \ \left(\mathrm{im}\left[\mathscr{B}^k(M,\partial M) \xrightarrow{\text{“}\sum_{v\in W'}\text{”} x_v} \mathscr{B}^{k+1}(M)\right]\right) \ \cap \ \mathscr{A}^k(\mathrm{st}^\circ_w M)$$

in $\mathscr{A}^k(\mathrm{st}_w M)$ and

$$\mathscr{K}_W \ := \ x_w\alpha \ \subset \ \mathscr{A}^k(\mathrm{st}_w M)$$

are orthogonal to each other. It follows that

$$x_w\ker\alpha \ \cap \ \mathrm{im}\alpha \ = \ 0 \quad \text{in } \mathscr{B}^k(M,\partial M)$$

if and only if the biased pairing property holds for $\mathscr{K}_W$ in $\mathscr{A}^*(\mathrm{st}_w M)$ in degree k.

To finally show that also $(1) \Rightarrow (3)$, it suffices to show

Lemma 7.3. $\mathscr{I}_W$ *and* $\mathscr{K}_W$ *are orthogonal complements in* $\mathscr{A}^k(\mathrm{lk}_w\, M)$.

For this, it suffices to argue that their dimensions sum up to $\dim \mathscr{A}^k(\mathrm{st}_w M)$, which follows by the short exact sequence

$$0 \ \longrightarrow \ \mathscr{A}^k(\mathrm{st}_w M) \ \longrightarrow \ \mathscr{B}^{k+1}(M) \ \longrightarrow \ \mathscr{B}^{k+1}(M - w \subset M) \ \longrightarrow \ 0$$

and because

$$\ker\alpha \cap \ker x_w \quad \text{and} \quad \mathrm{im}\alpha \ + \ \mathrm{im} x_w$$

are orthogonal complements in $\mathscr{B}^{k+1}(M - w \subset M)$ and its Poincaré dual in $\mathscr{B}^k(M,\partial M)$ □

We can therefore extend the transversal prime property from W' to W if any of the above conditions hold. Note further that the task to verify the conditions in Theorem 7.2 is much simpler if we use the transversal prime property applied to W', which we did not assume for its proof (indeed, we could have chosen α in any other way). In that case, the kernel of α has a nice description by induction, and so does its image.

As we shall see, this has an especially nice description if $M - W'$ and

$$\bigcup_{v\in W'} \mathrm{st}_v M$$

are codimension zero manifolds, when the full force of Proposition 4.5 applies. Unfortunately, this is not always the case. But we can force it, as we will do in Section 8. Let us first introduce two notions that illustrate this idea.

7.4. **Weakly deconstructible spheres and balls.** We illustrate the perturbation approach by proving the Lefschetz property for a larger class of spheres (and balls), which nevertheless can be decomposed nicely enough. Again, as natural, "ball" here means $\mathbb{k}$-homology manifold with the $\mathbb{k}$-homology of a ball.

To define this, we say that for a 0-dimensional complex, the removal of any vertex is a valid **weak deconstruction step**. Hence, any 0-dimensional complex is weakly deconstructible.

Consider now a $(d-1)$-dimensional ball or sphere X, and a vertex v in it with link $Y = \mathrm{lk}_v\, X$. Consider X_v the subcomplex induced by all facets not containing v.

We say that the transition from X to X_v is a **weak deconstruction step** if

(1) X_v is a $(d-1)$-ball or empty, and
(2) $\mathrm{lk}_v\, X_v$ is a weakly deconstructible $(d-2)$-ball or sphere or empty and obtained from $\mathrm{lk}_v\, X$ by a sequence of weak deconstruction steps, and finally
(3) $\partial\, \mathrm{lk}_v\, X_v$ is a weakly deconstructible $(d-3)$-sphere if nonempty.

A ball or sphere is **weakly deconstructible** if it can be reduced to the empty complex by weak deconstruction steps.

Theorem 7.4. *Consider a ball or sphere X of dimension $d-1$, and a sequence of weak deconstruction steps $V = (v_1, \cdots, v_n)$. Consider further a generic Artinian reduction $\mathscr{A}^*(X)$ of the face ring of X, and a generic ℓ supported on the vertices (v_i). Let finally X_V denote the ball induced by the remaining facets.*

Then, for all $k \le {}^{d}\!/_{2}$, the map

$$\mathscr{A}^k(X, \partial X) \xrightarrow{\ \cdot \ell^{d-2k}\ } \mathscr{A}^{d-k}(X)$$

has kernel $\mathscr{A}^k(X_V, \partial X_V)$.

Proof. Following the general perturbation approach, it suffices to consider $V = (v)$, that is, a single weak deconstruction step. Consider the link Y of $v \in X$, a projection π to $\mathbb{k}^{d-1}$ along v and ϑ the height over that projection. We prove the theorem by induction on the dimension.

We have to establish that the pullback $\mathscr{J}$ of $\mathscr{A}^k(X, \partial X)$ to $\mathscr{A}^k(\pi Y, \pi \partial Y)$ via the cone lemma satisfies the appropriate HGL relation: the pairing

$$\begin{array}{rcccc} \mathrm{Q}_{\mathscr{J},k,\vartheta}: & \mathscr{J}^k & \times & \mathscr{J}^k & \longrightarrow & \mathscr{J}^{d-1} \\ & a & & b & \longmapsto & ab\vartheta^{d-2k-1} \end{array} \tag{10}$$

is nondegenerate in $\mathscr{A}^k(\pi Y, \pi \partial Y)$. Recall that, by assumption, $Y_v = \mathrm{lk}_v\, X_v$ is obtained from $Y = \mathrm{lk}_v\, X$ by a sequence of weak deconstruction steps, which weakly deconstructs as well. Let $W = (w_1, \cdots, w_m)$ and $W' = (w_1', \cdots, w_n')$ be the corresponding sequence of vertices. We can assume that ϑ in Y is supported in $W \cup W'$ only; after all, we only make the linear system more special, and making it more generic later does not lose us the desired Lefschetz properties.

We decompose further and set $\vartheta = \vartheta_W + \vartheta_{W'}$. Let us initially also define $\vartheta_\lambda = \vartheta_W + \lambda\vartheta_{W'}$ for a scalar λ. Let us consider ϑ_0 first.

We can decompose $\mathscr{J} = \mathscr{J}_W \oplus \mathscr{J}'_{W'}$, where

$$\mathscr{J}'_W = \mathscr{A}^k(\pi Y_v, \pi\partial Y_v) \text{ and } \mathscr{J}_W = \vartheta_0 \mathscr{A}^{k-1}(\pi Y, \pi Y_v \cup \partial Y).$$

This decomposition is orthogonal under $\mathrm{Q}_{\mathscr{J},k,\vartheta_0}$, and by induction assumption, $\mathrm{Q}_{\mathscr{J},k,\vartheta_0}$ is nondegenerate on $\mathscr{J}_W$ because

$$\mathscr{A}^{k-1}(\pi Y, \partial Y) \xrightarrow{\cdot\vartheta_0^{d-2k+1}} \mathscr{A}^{d-1-(k-1)}(\pi Y)$$

has kernel $\mathscr{A}^{k-1}(\pi Y_v, \pi\partial Y_v)$.

Moreover, we have that $\vartheta_{W'}$ induces an isomorphism

$$\mathscr{A}^k(\pi Y_v, \pi\partial Y_v) \xrightarrow{\cdot\vartheta_0^{d-2k-1}} \mathscr{A}^{d-1-(k)}(\pi Y_v).$$

if $d > 2k+1$. If $d = 2k+1$, this is just the biased pairing property, which is characterized in turn by the Lefschetz property on ∂Y_v which we know by induction.

Hence, the Kronecker Lemma 7.1 gives the desired nondegeneracy of the HGL pairing for $\vartheta_\lambda = \vartheta_W + \lambda\vartheta_{W'}$, and therefore for ϑ itself. □

While still rather restrictive, the next sections show that proving the Lefschetz property for weakly deconstructible spheres and balls is all that is needed to prove the Lefschetz properties for all manifolds.

7.5. **Feebly deconstructible manifolds.** The restriction to spheres and balls is not necessary, and one can in fact extend the approach of the previous section to a notion that, in a sense, works on all orientable manifolds.

To define this, we say that for a 0-dimensional complex, the removal of any vertex is a valid **feeble deconstruction step**. Hence, any 0-dimensional complex is feebly deconstructible.

Consider now a $(d-1)$-dimensional orientable manifold X, and a vertex v in it with link $Y = \mathrm{lk}_v X$. Consider X_v the subcomplex induced by all facets not containing v.

We say that the transition from X to X_v is a **feeble deconstruction step** if

(1) X_v is a $(d-1)$-manifold or empty, and
(2) $\mathrm{lk}_v X_v$ is a feebly deconstructible $(d-2)$-manifold or empty and obtained from $\mathrm{lk}_v X$ by a sequence of feeble deconstruction steps, and finally
(3) $\partial\, \mathrm{lk}_v X_v$ is a feeble deconstructible $(d-3)$-manifold if nonempty.

A manifold is **feebly deconstructible** if it can be reduced to the empty complex by feeble deconstruction steps.

We leave the following as an exercise, as we will not use it further. See also [AY21].

Theorem 7.5. *Any feebly deconstructible orientable manifold has the Hard Lefschetz property, that is, a generic Artinian reduction of the face ring has the Hard Lefschetz property with respect to a generic linear element.*

Because the barycentric subdivision of any manifold is feebly deconstructible (another exercise) this implies the Hard Lefschetz property for the barycentric subdivision of any manifold. We will not use it, but instead make a mild topological restriction that allows us to prove the full theorem.

8. Railway construction

The plan now should be clear. For simplicity, let us first assume that $\Sigma = M$ is a sphere of even dimension $2k-2$.

If there is a total order on the vertices of Σ so that $\mathrm{N}_i\Sigma$, the union $\bigcup_{v\in\{1,\cdots,i\}} \mathrm{st}_v\, \Sigma_i$, is "nice enough" for all i (which we will define later), then we apply the approach of Section 7.3 directly.

The issue is of course what happens if this is not the case: In this case, we embed Σ into its suspension $\mathrm{susp}\, \Sigma$, and realize it generically in $\mathbb{k}^{[2k]}$. The original Σ, geometrically, sits in $\mathbb{k}^{[2k-1]}$, so let us distinguish it by referring to it by its projection $\pi\Sigma$. Let ϑ denote the linear form encoding the coordinates in $\mathbb{k}^{\{2k\}}$.

Since we want to prove the isomorphism

$$\mathscr{A}^{k-1}(\pi\Sigma) \xrightarrow{\ \cdot\vartheta\ } \mathscr{A}^{k}(\pi\Sigma)$$

we only care about the k-skeleton of Σ, which we denote by S for the moment. In $\mathrm{susp}\, \Sigma$, this simply means showing $\mathscr{A}^k(S) = 0$.

Assume now, after subdivisions that do not affect $S \subset \mathrm{susp}\, \Sigma$, the subcomplex S has an envelope E which is a manifold and *does* have a "nice enough" decomposition. Then we can apply Section 7.3. Of course, to say something about the biased pairing property at S, we want to prove a statement about the double of E, and apply Theorem 6.17. But that, it turns out in Section 9, is just a Lefschetz property for E itself.

Before we introduce it, let us pin down what we mean by a "nice enough" decomposition, which locally incorporates the notion of weak deconstructible balls and spheres. And before that, let us introduce a construction, which we will call the "Many Waists" Construction, and seems in itself an interesting observation in geometric topology.

8.1. **Many waists.** Consider a $(k-1)$-dimensional subcomplex Δ of a sphere Σ of dimension $d \geq 2k$. One curious question one might ask, and quite useful in the following, is whether there is a hypersphere (a hypersurface that is also a sphere), after some subdivision not affecting Δ in Σ containing Δ. That, of course is related to constructing the envelope we alluded to earlier, and a local necessity. Further, it is clear that the bound on dimension is tight. And were we in a PL sphere, then the answer would be yes, and the exercise is easy.

In the context of homology manifolds, this may sound much harder. And indeed, it is a little subtle.

A $\mathbb{k}$**-equivalence** $X \sim_{\mathbb{k}} X'$ **not affecting** $Y \subset X, X'$ is a $\mathbb{k}$-equivalence $X \sim_{\mathbb{k}} X'$ such that every $\mathbb{k}$-homeomorphism of the equivalence is the identity on the specified subcomplex Y. A $\mathbb{k}$-**deformation of a complex** X is a complex $\mathbb{k}$-equivalent to X.

Here is the construction, which we call "Many Waists" because finding the hypersphere (the "waist") is actually done quite flexibly:

The Many Waists Construction 8.1. Consider the biased derived subdivision $\varSigma'$ of $\varSigma$ at the $(k-1)$-skeleton $D \supset \varDelta$ of $\varSigma$. Let C denote the faces of dimension at most $d-(k+1)$ in $\varSigma'$ not intersecting $D \supset \varDelta$ of $\varSigma$. It is obviously $(d-k-2)$-acyclic by Alexander duality. Remove some $(d-k-1)$-dimensional faces until it is acyclic. Let C' be the result. Consider an ε-neighborhood around it[7]. The boundary B of that neighborhood is a sphere of dimension $(d-1)$.

We can retract it to D, and note that a generic fiber can be described explicitly: for every face σ of D, the fiber is of the form $(\mathrm{lk}_\sigma \Sigma' \cap C') \times \sigma$, and they are attached along the natural inclusions in D prescribed by the second factor.

Moreover, they are clearly $(d-k-3)$-acyclic, and we note $(d-k-3) \geq k-3$ by assumption. Hence, by contracting suitable acyclic subcomplex of $(\mathrm{lk}_\sigma \Sigma' \cap C')$ and therefore contracting suitable acyclic subsets of $B \subset \Sigma'$, we obtain an embedding of D (and therefore $\varDelta$) into a $(d-1)$-sphere $B' \subset \Sigma''$, which are $\mathbb{k}$-equivalent to B resp. Σ' because acyclic sets were contracted.

This construction is quite flexible: if $\varGamma \subset \varDelta$ is any subcomplex, then one can arrange a d-ball Q in $\varSigma$ that contains $\varGamma$ (this is a consequence of the construction itself), but no other face of $\varDelta$ because $d \geq 2k$. Choosing C' to be relatively acyclic in $(Q, \partial Q)$, and acyclic in its complement when performing the above construction yields a ball P in the hypersphere B' that contains $\varGamma$, but no other faces of $\varDelta$.

Corollary 8.2. *Consider a $(k-1)$-dimensional subcomplex $\varDelta$ of a sphere $\varSigma$ of dimension $d \geq 2k$. Then there is a $\mathbb{k}$-deformation of $\varSigma$ not affecting the $\varDelta$ that contains a hypersphere containing $\varDelta$. Moreover, if $\varGamma \subset \varDelta$ is any subcomplex, then one can arrange this waist in such a way that there is a $(d-1)$-ball contained in said hypersphere that contains $\varGamma$ but no other faces of $\varDelta$.*

One can easily see from here that the triangulation of the hypersphere can be chosen weakly deconstructible. We will skip it here, as we prove a more refined result later (Lemma 8.5).

8.2. **Tracks.** We differ here from [Adi18] with a different, and simpler construction.

First, let us define an appropriate notion of decomposition.

To define this, we say that for a 0-dimensional complex, the removal of any vertex is a valid **track-layer**. Hence, any 0-dimensional complex is a valid **track**.

A manifold pair (M, N) is a manifold M, and a codimension one submanifold of its boundary N, where N is allowed to be empty.

[7] Or simplicially, consider a further biased derived subdivision and the simplicial closure of all faces intersecting C' after this subdivision.

Consider now a $(d-1)$-dimensional orientable manifold X, B a $(d-2)$-dimensional submanifold of its boundary or empty, and a vertex v in (X, B) with link $\mathrm{lk}_v X$. Consider X_v the subcomplex induced by all facets not containing v.

We say that the transition from X to X_v is a **tracklaying** if

(1) X_v is a $(d-1)$-manifold or empty, $X_v \cap B$ a $(d-2)$-dimensional submanifold of its boundary (or empty) so that the inclusion of $(X_v, X_v \cap B)$ into (X, B) induces a surjection in cohomology in all dimensions $\leq {(d-1)}/{2}$, and
(2) $\mathrm{lk}_v X_v \cup (B \cap \mathrm{lk}_v X)$ is a track $(d-2)$-manifold or empty and obtained from $\mathrm{lk}_v X$ by a sequence of tracklaying steps, and finally
(3) $\partial(\mathrm{lk}_v X_v \cup (B \cap \mathrm{lk}_v X))$ is a track of dimension $(d-3)$ if nonempty.

The vertex v is a **bolt**.

The manifold pair (X, B) is a **track** if it can be reduced to the empty complex by tracklaying steps, and for every initial sequence of bolts $\{1, \cdots, i\}$, we have that

(4) The inclusion of $(\mathrm{N}_i X, B)$ into $(\mathrm{N}_{i+1} X, B)$ induces a surjection in cohomology in all dimensions $\leq {(d-1)}/{2}$.

The cohomological conditions are of course the difference, settling this notion between weak deconstructions and feeble ones.

8.3. **Métros.** We say an orientable manifold U in $\mathbb{k}^d$ with induced boundary and of dimension $d-1$ is a **métro** if

(1) U is a track,
(2) $(U, \partial U)$ is a track.

We call it a **railway in degree** k if, for some subcomplex Δ and under some geometric realization, $\mathscr{A}^k(U) = \mathscr{A}^k(\Delta)$. We call Δ its **tabula**.

It is often useful to impose an additional condition of topological simplicity. We call a railway resp. métro **octavian** if U has nontrivial cohomology at most in dimension $k-1$.

Let $U_I = \mathrm{N}_I U = \bigcup_{i \in I} \mathrm{st}_i U$. Let $\overline{U}_I$ denote the subcomplex of U induced by the facets of U *not* in U_I. Notice that métro in degree k is **hereditary**: for every initial segment W with final vertex w, and $W' = W \setminus (w)$ of tracklaying steps,

(1) the complex $\mathrm{lk}_w \overline{U}_{W'}$ is a track, of which $\overline{U}_W \cap \mathrm{lk}_w \overline{U}_{W'}$ is an intermediate state, that is, after some tracklaying steps we arrived at the latter before the complex is dismantled entirely, and
(2) $(\overline{U}_W \cup \partial U) \cap \mathrm{lk}_w \overline{U}_{W'}$ is an intermediate state (not necessarily of the same tracklaying sequence).

Notice that the first condition follows because U is a track, and that the second follows because $(U, \partial U)$ is a track. We just state it again here for ease of reference. Also, notice that the tracklayings for U and $(U, \partial U)$ do not have to coincide, though the construction we will provide does have this property.

Lemma 8.3. *Consider a $(k-1)$-dimensional subcomplex S of a d-sphere Σ of dimension at least $2k-1$. Then there is a $\mathbb{k}$-equivalence $\Sigma \sim_{\mathbb{k}} \widehat{\Sigma}$ not affecting S such that $\widehat{\Sigma}$ contains a hereditary octavian métro U of dimension $d-1$.*

Moreover, U can be chosen so that there is $S' \subset \Sigma \subset \widehat{\Sigma}$ with $S' \supset S$ a complex of dimension k so that U collapses to S'.

This is, in a sense, a variant of the Many Waists Construction; for a $(k-1)$-dimensional complex in a $(2k-1)$-dimensional sphere, we cannot hope to embed it into a hypersphere. A track is the best we can hope for.

We need some combinatorial topology. Inspired by Whitehead's notion of shellability [Whi38], we say that a $(d-1)$-manifold M **elementarily** W**-shells** to a manifold M' if $M = M' \cup D$ and $M' \cap D$ is a $(d-2)$-ball or sphere. A W**-shelling** is a composition of several elementary W-shelling steps. We denote simplicial collapses by $\searrow$, and W-shellings by $\searrow_W$.

Lemma 8.4. *Consider any k-dimensional subcomplex Δ of a d-sphere Σ as a subcomplex of* $\operatorname{susp} \Sigma$, *$k+1 < d$ or $k=0, d=1$, and assume the vertices of Δ are totally ordered.*

Let Δ_i denote the subcomplex $\mathrm{N}_i \Delta = \bigcup_{v \in \{1,\cdots,i\}} \mathrm{st}_v\, \Delta_i$ induced by neighborhoods of the first i vertices. Then there is a $\mathbb{k}$-equivalence $\widehat{\Sigma} \sim_{\mathbb{k}} \operatorname{susp} \Sigma$ of $\operatorname{susp} \Sigma$ that does not affect Σ, and D_i d-dimensional submanifolds of $\widehat{\Sigma}$, such that for all i,

(1) D_i is a ($\mathbb{k}$-homology) ball containing Δ_i, but no other faces of Δ, and
(2) D_{i+1} W-shells to D_i, and
(3) the final D_{final} is a sphere.

Moreover, we can arrange this W-shelling to be a weak deconstruction. Additionally, if the sequence is prescribed up to a D_j and this D_j lies in Σ, then it may be continued from there.

Proof. The case $k=0, d=1$ is a simple exercise, so we focus on the case $k+1<d$.

Consider one of the suspension vertices, say $\mathbf{n}$. We consider the cones $C_i = \mathbf{n} * \Delta_i$ in $\operatorname{susp} \Sigma$. We have a sequence of simplicial collapses

$$\cdots \searrow C_i \searrow C_{i-1} \searrow \cdots \searrow C_1 \searrow \mathbf{n}$$

These are contractible, and therefore their topological neighborhoods B_i are balls of dimension $d+1$. After subdivisions not affecting Σ we can arrange these neighborhoods B_i in an ascending way, that is

$$\cdots \searrow_W B_i \searrow_W B_{i-1} \searrow_W \cdots \searrow_W B_1 \searrow \searrow_W B_0$$

and so that they contain Δ_i in the boundary, but no other faces of Δ. We can refine this further into elementary W-shelling moves, let us denote the sequence by

$$\cdots \searrow_W B'_j \searrow_W B'_{j-1} \searrow_W \cdots \searrow_W B'_1 \searrow \searrow_W B'_0 = B_0$$

In each of these cases, $B'_j = F_j \cup B'_{j-1}$, where F_j is a $(d+1)$-ball that intersects B'_{j-1} in a d-ball K'_j. Let K_j be the complementary d-ball in ∂F_j. Because $d-k>1$, we can assume that the homology $(d-1)$-spheres ∂K_i share a common point p.

Remove a small neighborhood P of that point. Then, if the elementary W-shelling ends with B'_m, we set $D'_m = \partial B'_m \setminus U$. We find $D'_{m-1} = D'_m \setminus K_m$, and so forth. The eventual D_{final} is obtained by adding P back again to D'_m. Iterating in dimension gives the weak deconstruction property. The prescription of initial segments is clear from the construction. □

We obtain a refinement of the Many Waists Construction 8.1.

Lemma 8.5. *Consider any k-dimensional subcomplex Δ of a d-sphere Σ, $2k+1 < d$, and assume the vertices of Δ are totally ordered.*

Let Δ_i denote the subcomplex $\mathrm{N}_i\Delta = \bigcup_{v\in\{1,\cdots,i\}} \mathrm{st}_v\, \Delta_i$ induced by neighborhoods of the first i vertices. Then there is a $\Bbbk$-equivalent $\widehat{\Sigma} \sim_{\Bbbk} \Sigma$ of Σ that does not affect Δ, and D_i d-dimensional submanifolds of $\widehat{\Sigma}$, such that for all i,

(1) all but the final D_i is a ($\Bbbk$-homology) ball containing Δ_i, but no other faces of Δ, and
(2) D_{i+1} W-shells to D_i, and
(3) the final D_{final} is a sphere.

Moreover, we can arrange this W-shelling to be a weak deconstruction. Additionally, if the sequence is prescribed up to a D_j, it may be continued from there.

Proof. Use the Many Waists Construction 8.1, we can find a $\Bbbk$-equivalence not affecting Δ to a suspension $\mathrm{susp}\,\overline{\Sigma}$ so that Δ is k-dimensional subcomplex Δ of a $(d-1)$-sphere $\overline{\Sigma}$. Apply Lemma 8.4.

If an initial segment is prescribed, construct $\overline{\Sigma}$ so that D_j is a ball in it, which is easily done by contracting D_j to a point before applying the Many Waists Construction, and then expanding it back afterwards. □

We obtain at once:

Proof of Lemma 8.3. Order the vertices of S in an arbitrary way. We want to modify S in such a way that S_i, the subcomplex $\mathrm{N}_i S = \bigcup_{v\in\{1,\cdots,i\}} \mathrm{st}_v\, S_i$, is $(k-2)$-acyclic, and so that the complex $\overline{S}_i$ induced by the faces not in S_i is $(k-2)$-acyclic as well. We do so as follows: Consider the polyhedral complex $\Sigma \times P$, where P is a path of length three from vertices 1 via 0 to -1. The cone over Σ with apex $\mathbf{n}$, and attach it to $\Sigma \times P$ along $\Sigma \times \{1\}$. Similarly with $\mathbf{s}$ and $\Sigma \times \{-1\}$, therefore obtaining a bigger polyhedral complex $\widetilde{M}$ that is $\Bbbk$-equivalent to $\mathrm{susp}\,\Sigma$ in a way that does not affect $\Sigma \times \{-1\}$. Consider a maximal, $\Bbbk$-acyclic subcomplex to $\Sigma \times \{1\}$ that contains all $(k-1)$-dimensional faces. After sufficiently many stellar subdivisions of $\Sigma \times \{1\}$, we can contract it to a point in a map of polyhedra. Call the point s_0. Similarly with $\Sigma \times \{-1\}$, with a vertex s_{final}. The new complex S' is obtained as S, union with the cone with apex s_0 and base the $(k-2)$-skeleton of S, union with the cone with apex s_{final} and base the $(k-1)$-skeleton of S. In the order of vertices, s_0 precedes all vertices of S, and s_{final} follows them. That takes care of $(k-2)$-acyclicity.

We now construct the track along the vertices s_* as its bolts. We start with s_0. Using Lemma 8.5, we find a $\Bbbk$-equivalence not affecting S' and a $(2k-3)$-sphere in the link of s_0 containing $\mathrm{lk}_{s_0} S'$. In the thus resulting $\Bbbk$-equivalent deformation of $\widetilde{M}$, this corresponds to a $(2k-2)$-ball U_0. Note that we can arrange $\mathrm{lk}_{s_0} S'$ to be weakly decomposable, and hence a track.

We continue with vertices in S. Consider next vertex s_i, assuming all prior vertices of S' have been added. Then $\mathrm{lk}_{s_i} U_{i-1}$, where $U_{i-1} = U_{\{s_0,s_1,\cdots s_{i-1}\}}$ is a $(2k-3)$-ball containing some, but not all faces of $\mathrm{lk}_{s_i} S'$. Using Lemma 8.5 we extend it to a $(2k-1)$-sphere that contains all of them, applying it to a $\Bbbk$-equivalent deformation where $\mathrm{lk}_{s_i} U_{i-1}$ was contracted to a point. This construction of course gives a weak deconstruction. Allowing for the desired initial segments while weakening to a track is straightforward.

After sufficiently many refinements outside S, we can make sure the neighborhood S_i' is almost a manifold: The link of every face in S_i is a sphere, and everywhere else it behaves like an open neighborhood that collapses onto S_i. The only place where an issue can arise is a vertex of S' in S_i, but is not among its first i vertices.

The link $\mathrm{lk}_s U_i$ of such a vertex, say s, can fail to be a ball. However, this is repaired by Lemma 8.5: in each step, $\mathrm{lk}_s U_j$ is a manifold that collapses to $\mathrm{lk}_s S_j'$. Hence we can find a $(2k-3)$-ball containing it, as $\mathrm{lk}_s S_j'$ can be enveloped by a $(2k-3)$-ball (again using the Many Waists Construction), but no other faces of $\mathrm{lk}_s S'$. Hence it also contains $\mathrm{lk}_s U_j$ after $\Bbbk$-equivalence.

After these modifications, U_i is a ball in the link of s, and therefore we made sure U_i is a manifold.

The octavian property is easily ensured as sufficiently small neighborhoods collapse onto the nerve, which justifies passing from S to S', and the homological conditions of the track are ensured because we passed from S to S' in the beginning. □

8.4. **Métros to railways.** It remains to ensure the envelope property to transition from métros to railways.

Proposition 8.6. *Consider a $(k-1)$-dimensional subcomplex S of a $(2k-2)$-sphere Σ as a subcomplex of $\mathrm{susp}\,\Sigma$. Then there is a $\Bbbk$-equivalence $\mathrm{susp}\,\Sigma \sim_{\Bbbk} \widehat{\Sigma}$ of $\mathrm{susp}\,\Sigma$ that doesn't affect Σ and contains an octavian railway in degree k for tabula S.*

For the proof, we only need the following final lemmata, generalizing the ideas of Corollary 6.11:

Lemma 8.7. *Consider a $(k-1)$-dimensional subcomplex Δ of a $(2k-1)$-sphere Σ. Assume E is a hypersurface of Σ containing Δ, and that E has a simplicial collapse onto Δ. Then some subdivision E' of E that doesn't affect Δ is an envelope for Δ in degree k if the geometric realization (or equivalently the linear system of parameters) is chosen generically.*

Proof. Consider an elementary collapse $E \searrow_e E'$, along a free face σ. Stellarly subdivide at σ, and move the new vertices into general position.

By the weak Lefschetz theorem for $\mathrm{lk}_{e_\sigma} E{\uparrow}\sigma$ (which holds as this is just projective space, that is, the face ring of a simplex), (where e_σ is the unique interior edge to $E{\uparrow}\sigma$ incident to v_σ), we see that after the subdivision, the open star $\mathrm{st}^\circ_{v_\sigma} E{\uparrow}\sigma$ does not support a degree k element: If ϑ is the height over a general position hyperplane in $v^\perp \in \Bbbk^d$, π the projection to that hyperplane, then

$$\mathscr{A}^{k-2}(\pi\,\mathrm{lk}_{v_\sigma} E{\uparrow}\sigma) \xrightarrow{\cdot\vartheta} \mathscr{A}^{k-1}(\pi\,\mathrm{lk}_{v_\sigma} E{\uparrow}\sigma)$$

is a surjection.

In particular, $\mathscr{A}^k(\mathrm{st}^\circ_{v_\sigma} E{\uparrow}\sigma)$ is zero by the cone lemma. Hence, any degree k element of $E{\uparrow}\sigma$ must be supported in E'.

Figure 8.3. Subdividing a simple homotopy collapse along a free edge e, and perturbing it to remove stresses in the neighborhood of the newly created vertex v_e

Repeating this we see that after stellar subdivisions along the collapses, we observe that any degree k element of the railway can be assumed to be generated by the faces of Δ. □

The other lemma deals with the fact that our tabula is S', not S. This again is easily dealt with.

Lemma 8.8. *Consider a $(k-1)$-dimensional subcomplex Δ of a $(2k-1)$-sphere Σ. Assume Δ' is a $(k-1)$-dimensional subcomplex of Σ containing Δ. Then some subdivision Δ'' of Δ' that doesn't affect Δ is an envelope for Δ in degree k if the geometric realization (or equivalently the linear system of parameters) is chosen generically.*

Proof. We perform stellar subdivisions at every $(k-1)$-dimensional face of Δ' not in Δ. Using the Lefschetz property of the simplex again, we obtain the desired. □

8.5. **Railways in manifolds.** Let us briefly note the more general theorem:

Proposition 8.9. *Consider a k-dimensional subcomplex S of a $2k$-manifold M as a subcomplex of $\operatorname{susp} M$. Then there is a $\Bbbk$-equivalence $\operatorname{susp} M \sim_{\Bbbk} \widehat{M}$ of $\operatorname{susp} M$ that doesn't affect S and contains a hereditary octavian railway for S.*

Here, the acyclicity in the octavian condition is replaced by relative acyclicity in

$$\operatorname{susp} M \setminus \{\text{suspension points}\} \simeq M.$$

The proof is essentially identical, and can be skipped here.

9. Putting things together

Now, again, we put things together. We first treat the case of spheres.

9.1. **Revisiting the characterization theorem.** We return to Section 6.8, but restrict to the case when M is a sphere of dimension $2k-1$. Then E is a $(k-2)$-acyclic orientable hypersurface with boundary.

Lemma 9.1. *Assume that ϑ is supported on the interior vertices of E, and*

$$\mathscr{B}^{k-1}(E) \xrightarrow{\ \cdot\vartheta\ } \mathscr{B}^{k}(\pi E, \pi\partial E)$$

is an injection.

Then the map

$$(H^{k-1})^{\binom{d}{k}}(\widetilde{\mathrm{D}}M) \longrightarrow \mathscr{A}^k(\mathrm{D}E) \tag{11}$$

is an injection.

Proof. $\mathrm{D}E$ is realized in $\Bbbk^{[d]}$. We consider $\pi\mathrm{D}E$, realized in $\Bbbk^{[d-1]}$. Then we have an injection

$$(H^{k-1})^{\binom{[d-1]}{k}}(\widetilde{\mathrm{D}}M) \hookrightarrow (H^{k-1})^{\binom{[d-1]}{k}}(\mathrm{D}E) \hookrightarrow \mathscr{A}^k(\pi\mathrm{D}E).$$

The cokernel of the right-hand map is $\mathscr{B}^k(\pi\mathrm{D}E)$. Notice that the kernel of the map

$$\tau^* : H_{k-1}(\mathrm{D}E) \longrightarrow H_{k-1}(E)$$

generates $H_{k-1}(\widetilde{\mathrm{D}}M)$, hence $\hom$ embeds $(H^{k-1})^{\binom{[d-1]}{k}}(\widetilde{\mathrm{D}}M)$ into

$$\mathscr{A}^k(\mathrm{D}E) = \mathscr{A}^k(\pi\mathrm{D}E)\Big/\vartheta\mathscr{A}^{k-1}(\pi\mathrm{D}E)$$

and

$$\mathscr{A}^k(\mathrm{D}E)\Big/(H^{k-1})^{\binom{[d-1]}{k}}(\widetilde{\mathrm{D}}M) \twoheadrightarrow \mathscr{A}^k(\mathrm{D}E)\Big/(H^{k-1})^{\binom{[d-1]}{k}}(\mathrm{D}E) \cong \mathscr{B}^k(\pi\mathrm{D}E)\Big/\mathscr{B}^{k-1}(\pi\mathrm{D}E)$$

It therefore remains to construct an embedding

$$(H^{k-1})^{\{d\}\cup\binom{[d-1]}{k-1}}(\widetilde{\mathrm{D}}M) \hookrightarrow \mathscr{B}^k(\pi\mathrm{D}E)\Big/\mathscr{B}^{k-1}(\pi\mathrm{D}E).$$

To this end, let us label the two sheets of E in $\mathrm{D}E$ by E_0 and E_1. Consider the subspace $\mathscr{H}$ of elements in $\mathscr{A}^{k-1}(\pi E) = \mathscr{B}^{k-1}(\pi E)$ whose restriction to $\mathscr{A}^{k-1}(\pi\partial E)$ lies in $\hom(H^{k-1})^{\binom{[d-1]}{k-1}}(\partial E)$.

Now, ϑ maps $\mathscr{H} \subset \mathscr{A}^{k-1}(\pi E)$ injectively to $\mathscr{A}^k(\pi E, \pi\partial E)$ by assumption.

Consider therefore an $\alpha \in \mathscr{H}$. We have two copies, represented as $\alpha_0 \in \mathscr{H}_0 \subset \mathscr{A}^{k-1}(\pi E_0)$ and $\alpha_1 \in \mathscr{H}_1 \subset \mathscr{A}^{k-1}(\pi E_1)$, respectively, mapped to $\mathscr{A}^k(\pi E_0, \pi\partial E_0)$ and $\mathscr{A}^k(\pi E_1, \pi\partial E_1)$ by ϑ_0 and ϑ_1, respectively.

We map α to

$$\vartheta_0\alpha_0 - \vartheta_1\alpha_1 \in \mathscr{A}^k(\pi E_0, \pi\partial E_0) \oplus \mathscr{A}^k(\pi E_1, \pi\partial E_1),$$

which in turn maps to $\mathscr{A}^k(\pi\mathrm{D}E)$.

The image under this map is clearly in the cokernel of

$$\mathscr{B}^{k-1}(\pi\mathrm{D}E) \longrightarrow \mathscr{B}^k(\pi\mathrm{D}E)$$

Moreover, the image $\widetilde{\mathscr{H}}$ coincides with the image of $(H^{k-1})^{\{d\}\cup\binom{[d-1]}{k-1}}(\widetilde{\mathrm{D}}M)$ in

$$\mathscr{B}^k(\pi\mathrm{D}E)\Big/\mathscr{B}^{k-1}(\pi\mathrm{D}E).$$

Combining this with the earlier embedding

$$(H^{k-1})^{\binom{[d-1]}{k}}(\widetilde{\mathrm{D}}M) \hookrightarrow \mathscr{A}^k(\pi\mathrm{D}E)$$

this gives the desired. $\square$

9.2. **Two Lefschetz properties for a manifold with boundary.** Now, let us assume the conditions of Lemma 6.16 hold, that is,

$$\mathscr{B}^{k-1}(\pi E, \partial\pi E) \xrightarrow{\cdot\overline{\vartheta}} \mathscr{A}^k(\pi E)$$

is an injection.

Then, invoking Theorem 6.17, we see that Map (11) is injective if and only if

$$\mathscr{K}^k(\widetilde{\mathrm{D}}M, \mathrm{D}E) \longrightarrow \mathscr{B}^k(\widetilde{\mathrm{D}}M)$$

is injective, which in turn implies the biased pairing property for $\mathscr{K}^k(M, E)$.

So, we observe that we need to prove the first property for our purposes.

Hence, to summarize, to prove the biased pairing property for $\mathscr{K}^k(M, E)$, one has to establish

(1) an injection
$$\mathscr{B}^{k-1}(\pi E, \partial\pi E) \xrightarrow{\cdot\overline{\vartheta}} \mathscr{A}^k(\pi E)$$
and

(2) an injection
$$\mathscr{B}^{k-1}(E) \xrightarrow{\cdot\vartheta} \mathscr{B}^k(\pi E, \pi\partial E)$$
where ϑ is supported on the interior vertices of E.

Note that $\overline{\vartheta}$ and ϑ do not have to coincide. In particular, $\overline{\vartheta}$ does not have to be supported on interior vertices. Because if we have proven the property (2), then we have proven that Map (11) is injective for that particular ϑ, and hence for every generic ϑ and without having to restrict to interior vertices only.

Note further that these are properties of E, not its double. The latter has a very non-generic linear system of parameters, and proving anything for that case using our techniques is very cumbersome. The system of parameters on E, instead, is rather generic (apart from the condition that ϑ is supported on interior vertices.)

Moreover, the map of (2) is one between Poincaré dual spaces, so it is an injection if and only if it is an isomorphism of Lefschetz type. The first, while not a Lefschetz map as such, can be strengthened to

(1)′ an injection
$$\mathscr{B}^{k-1}(\pi E, \partial\pi E) \xrightarrow{\cdot\vartheta} \mathscr{B}^k(\pi E)$$
and

in which case it, too, is a map between dual spaces and hence equivalent to a Lefschetz isomorphism.

To make it clear then what remains to prove Theorem 1.1 for spheres is to establish these two isomorphisms for octavian, hereditary railways.

9.3. **Lefschetz for spheres via Lefschetz for railways via Lefschetz for tracks.** Consider now a hereditary, octavian railway E in a sphere Σ of dimension $2k-1 = d-1$ realized in $\mathbb{k}^{[}d]$, which comes with a total order on interior vertices $(v_1, \cdots v_n)$. Our aim is to prove the following:

Lemma 9.2. *Assume the linear system on Σ is in sufficiently general position in $\Bbbk^{[d-1]}$. Then, for generic $\overline{\vartheta}$ and ϑ, the latter supported on the interior vertices of E, we have isomorphisms*

(1)
$$\mathscr{B}^{k-1}(\pi E, \partial\pi E) \xrightarrow{\cdot\overline{\vartheta}} \mathscr{B}^{k}(\pi E)$$
(2)
$$\mathscr{B}^{k-1}(E) \xrightarrow{\cdot\vartheta} \mathscr{B}^{k}(\pi E, \pi\partial E).$$

This in particular implies, with the preceding discussion, Theorem 1.1 for spheres of dimension $2k-2$. Of course, Lemma 9.2 is itself the consequence of the following, cleaner lemma:

Lemma 9.3. *Consider U an orientable $(2k-2)$-manifold, and let $T = \partial U$ or $T = \varnothing$. Let $\overline{T} = \partial U \setminus T$. Assume (U, T) is a track. Then U has the Lefschetz property in degree k, that is, for a generic realization of U in $\Bbbk^{[2k-1]}$ and ℓ a generic element of $\mathscr{A}^1(U)$ supported on the bolts, we have an isomorphism*
$$\mathscr{B}^{k-1}(U, T) \xrightarrow{\cdot\ell} \mathscr{B}^{k}(U, \overline{T}).$$

Proof. Following the general perturbation approach, it suffices to consider an initial segment W of the bolts, and let w be the final vertex of W, and $W' = W \setminus (w)$. We assume by induction that
$$\mathscr{B}^{k-1}(U, T) \xrightarrow{\cdot\ell_{W'}} \mathscr{B}^{k}(U, \overline{T})$$
has kernel $\mathscr{A}^{k-1}(U, T \cup \mathrm{N}_{W'}U) \subset \mathscr{B}^{k-1}(U, T)$ (that is, the kernel is the image of the former in the latter). It remains to show that the kernel pulled back to $\mathscr{A}(\pi \operatorname{lk}_w U)$, has the biased pairing property in degree $k-1$. This is proven as in Theorem 7.4:

Let $\mathscr{J}$ denote the pullback of the kernel to $\mathscr{A}^{k-1}(\pi \operatorname{lk}_w U)$, where π is the projection along w to $\Bbbk^{[2k-2]}$ and ϑ the remaining linear form. Recall that, by assumption, $\operatorname{lk}_w \overline{U}_{W'}$ is a track, of which $Y = (\overline{U}_W \cup T) \cap \operatorname{lk}_w \overline{U}_{W'}$ is an intermediate state (that is, after some tracklaying steps, we arrived at this subcomplex). Let $\underline{W} = (\underline{w}_1, \cdots, \underline{w}_m)$ and $\overline{W} = (\overline{w}_1, \cdots, \overline{w}_n)$ be the corresponding sequence of vertices. We may assume that ϑ in $\operatorname{lk}_w \overline{U}_{W'}$ is supported in $\underline{W}$ only (after all, we are making the system just more special).

Then we can decompose $\mathscr{J} = \mathscr{J}_{\underline{W}} \oplus \mathscr{J}_{\overline{W}}$, where $\mathscr{J}_{\overline{W}}$ is the image of $\mathscr{A}^{k-1}(\pi Y, \pi\partial Y)$ in $\mathscr{A}^{k-1}(\pi \operatorname{lk}_w U)$ and $\mathscr{J}_{\underline{W}}$ is the image of $\vartheta\mathscr{A}^{k-2}(\pi \mathrm{N}_{\underline{W}} \operatorname{lk}_w \overline{U}_{W'}, \pi\partial(\mathrm{N}_{\underline{W}} \operatorname{lk}_w \overline{U}_{W'}))$ in $\mathscr{A}^{k-1}(\pi \operatorname{lk}_w U)$. This decomposition is orthogonal under Q, and one sees by induction that Q is nondegenerate on $\mathscr{J}_{\underline{W}}$ because
$$\mathscr{A}^{k-2}(\pi \operatorname{lk}_w \overline{U}_{W'}, \pi\partial \operatorname{lk}_w \overline{U}_{W'}) \xrightarrow{\cdot\vartheta^2} \mathscr{A}^{k}(\pi \operatorname{lk}_w \overline{U}_{W'}).$$
has kernel $\mathscr{A}^{k-1}(Y, \partial Y)$.

Moreover, we have that the pairing is nondegenerate on $\mathscr{J}_{\overline{W}}$ because the boundary of Y is a track by assumption. Hence, if ℓ_W is the generic linear combination of $\ell_{W'}$ and x_w, we have that
$$\mathscr{B}^{k-1}(U, T) \xrightarrow{\cdot\ell_{W}} \mathscr{B}^{k}(U, \overline{T})$$
has kernel $\mathscr{A}^{k-1}(U, T \cup \mathrm{N}_W U) \subset \mathscr{B}^{k-1}(U, T)$. Iterating gives the desired isomorphism. $\square$

9.4. **Passing to manifolds.** For manifolds, the same approach works, but we have to pass to relative homology and acyclicity, with a caveat: The suspension is not a manifold, and we have to determine the socle in terms of the homology of the manifold M.

Theorem 9.4. *Let πM be a manifold of dimension $d-1$, realized in $\Bbbk^{[d]}$, and consider the suspension* $\operatorname{susp} \pi M$ *realized in $\Bbbk^{[d+1]}$. Assume that the suspension points project to $\mathbf{0}$ in $\Bbbk^{[d]}$.*

Then the kernel of

$$\mathscr{A}^k(\operatorname{susp} M) \longrightarrow \bigoplus_{v\in M^{(0)}} \mathscr{A}^k(\operatorname{st}_v \operatorname{susp} M)$$

is isomorphic to

$$(H^{k-1}(M))^{\binom{[d]}{k}} \oplus (H^{k-2}(M))^{\{d+1\}\cup\binom{[d]}{k-1}}.$$

Proof. Consider north pole $\mathbf{n}$ and south pole $\mathbf{s}$. This follows from the short exact sequence

$$0 \longrightarrow \mathscr{A}^*(\operatorname{susp} M, \operatorname{st}_{\mathbf{s}} \operatorname{susp} M) \longrightarrow \mathscr{A}^*(\operatorname{susp} M) \longrightarrow \mathscr{A}^*(\operatorname{st}_{\mathbf{s}} \operatorname{susp} M) \longrightarrow 0$$

The left and right side are, by cone lemma, isomorphic to $\mathscr{A}^*(\pi M)$, with the left side a copy shifted by one degree. □

Hence, once again, we obtain a description of the socle of the suspension; the ring $\mathscr{B}(\operatorname{susp} M)$ is defined as the quotient by the open socle.

The proof of Theorem 1.1 for closed, orientable manifolds follows analogously from here, reducing via railways to the case of tracks, which, as we saw, reduces to analyzing weakly deconstructible spheres. The case of orientable manifolds with boundary in Theorem 5.12 only needs minor adaptation to account for relative homology, and similarly analogous.

10. Further remarks and directions

Let us close this manuscript by first discussing the Hard Lefschetz properties.

10.1. **Hard Lefschetz and the HGL relations.** We now recall a technique of [Adi18] to establish the HGL relations and hard Lefschetz property, and in fact obtain yet another reduction of the Lefschetz property to biased pairing directly. We can restrict to orientable closed manifolds, which we close as a class under suspensions. These suspensions are chosen as in Section 9.4: If πX is realized in $\Bbbk^{[d]}$, then $\operatorname{susp} X$ is realized in $\Bbbk^{[d+1]}$ so that the suspension points project to the origin. We have $\mathscr{B}(X)$ defined as usual, the unique Gorenstein quotient of $\mathscr{A}(X)$ with the same top socle element.

To this end, assume we wish to prove the HGL relations for a pair (X, Y), where Y is a subcomplex of X a $(d-1)$-complex in the class, specifically the HGL relations for $\mathscr{K}^*(X, Y)$. Label the two vertices of the suspension $\mathbf{n}$ and $\mathbf{s}$ (for north and south). We then argue using the following observation, which follows as Proposition 6.7. We remind ourselves that $A * B$ denotes the free join of two simplicial complexes A and B.

Lemma 10.1. *Considering $\operatorname{susp} X$ realized in $\Bbbk^{d+1}$, and $k < \frac{d}{2}$, the following two are equivalent:*

(1) The HGL relations for

$$\mathscr{K}^{k+1}(\operatorname{susp} X, \operatorname{susp}(Y) \cup \mathbf{s} * X)$$

with respect to $x_{\mathbf{n}}$.

(2) The HGL relations for

$$\mathscr{K}^{k}(\pi X, \pi Y)$$

with respect to ϑ.

(3) The HGL relations for

$$\mathscr{K}^{k+1}(\operatorname{susp} X, \operatorname{susp}(Y) \cup X)$$

with respect to $x_{\mathbf{n}} - x_{\mathbf{s}}$.

Proof. Without loss of generality we have $\vartheta = x_{\mathbf{n}} - x_{\mathbf{s}}$ in $\mathscr{A}^*(\operatorname{susp} X)$. Consider then the diagram

$$\begin{tikzcd}
\mathscr{A}^{k}(\pi X) \arrow[r, "\cdot\vartheta^{d-2k}"] \arrow[d, "\sim"] & \mathscr{A}^{d-k}(\pi X) \arrow[d, "\sim"] \\
\mathscr{A}^{k+1}(\operatorname{susp} X, \mathbf{s} * X) \arrow[r, "\cdot x_{\mathbf{n}}^{d-2k-1}"] & \mathscr{A}^{d-k}(\mathbf{n} * X)
\end{tikzcd}$$

where the first vertical map is defined by the composition of cone lemmas

$$\mathscr{A}^{k}(\pi X) \ \cong\ \mathscr{A}^{k}(\mathbf{n} * X) \ \xrightarrow{\cdot x_{\mathbf{n}}}\ \mathscr{A}^{k+1}(\operatorname{susp} X, \mathbf{s} * X).$$

and the second vertical map is simply the cone lemma. An isomorphism on the top is then equivalent to an isomorphism of the bottom map, and restricting to ideals and their Poincaré duals gives the equality of the first two statements.

The first statement and the last statement are equivalent because

$$\mathscr{K}^{k+1}(\operatorname{susp} X, \mathbf{s} * X)$$

and

$$\mathscr{K}^{k+1}(\operatorname{susp} X, \mathbf{n} * X)$$

are orthogonal on each other. $\square$

Hence, one can inductively reduce the study to the weak Lefschetz property. The main final Theorem is then:

Theorem 10.2. *Fix an infinite field $\Bbbk$ of any characteristic. Consider a triangulated orientable compact $\Bbbk$-homology manifold M of dimension $d-1$, and the associated graded commutative face ring $\Bbbk[M]$. Then the Gorensteinification of a generic Artinian reduction $\mathscr{A}^*(M)$ of $\Bbbk[M]$ has the hard Lefschetz property with respect to a generic $\ell \in \mathscr{A}^1(M)$, that is, for every $k \le d/2$, an isomorphism*

$$\mathscr{B}^{k}(M, \partial M) \ \xrightarrow{\cdot \ell^{d-2k}}\ \mathscr{B}^{d-k}(M).$$

Moreover, if M is closed, then it has the HGL relations.

Proof. Let us note that the last part of this result, the HGL relations, implies all the previous parts (including the case of orientable manifolds with boundary). Now we notice that we already constructed railways in sufficient generality: If M is a manifold of dimension $d-1 = 2k-1$, and Δ

is any subcomplex, then there is an octavian railway E in degree k for Δ in a $\mathbb{k}$-deformation that does not affect Δ (our construction of railways already took this into account). The biased pairing property at Δ, that is, at the ideal $\mathscr{K}(M,\Delta)$ in $\mathscr{B}^*(M)$.

Now, the HGL relations in degree k for M of dimension $d-1$, $d>2k$ are of course reduced to the biased pairing property in iterated suspensions, but then to k' and d' with $d'=2k'$ by Lemma 10.1.

Indeed, we are then trying to prove, in $X=\operatorname{susp}^{d-2k} M$, the biased pairing property with respect to the ideal $\mathscr{K}(X,Y)$, where Y is the subcomplex

$$\operatorname{susp}^{d-2k} Y \cup \operatorname{susp}^{d-2k-1} X \cup \operatorname{susp}^{d-2k-2} X \cdots .$$

Instead of railways, we consider iterated suspensions over railways, and arrange the bolts by removing the suspension vertices first, starting with the last suspension performed. This reduces to railways on one fewer suspension etc.

If $d=2k+1$, this is an immediate reduction, as Lemma 10.1 induces no further conditions on the linear system of parameters. For $d>2k+1$, it is not hard to see that the inductive reduction still, ultimately, reduces to the fully generic Lefschetz property for tracks by examining the descent carefully. $\square$

10.2. **Limits of the theory: between pairings and numerical monotonicity.** [Adi18] introduced the *biased pairing property*, which leads to a proof of the Lefschetz property in arbitrary characteristic: It says that the Lefschetz property can be reduced to a nondegeneracy of Poincaré pairings when restricting to subspaces.

Papadakis and Petrotou introduced total anisotropy: the property that the Poincaré pairing does not degenerate at *any* ideal (or equivalently, at principal ideals) [PP20, APP23, APP25]. They then use the reduction of [Adi18] to prove the Lefschetz property.

That is, we have,

$$\text{total anisotropy} \rightsquigarrow \text{biased pairing property} \rightsquigarrow \text{Lefschetz property}$$

with the curiosity that the biased pairing property also leads back to the Lefschetz property, in precise ways (see our Section 6).

But in the setting of the Geiringer-Laman Theorem 5.4, and if we are permitted to mention combinatorics as well, the ubiquitous Hall's marriage theorem [Hal35], a simpler way also suffices: A numerical monotonicity property of ideals, that is implied by the Lefschetz property (see Observation 5.1) and biased pairing property as in Section 6.3.

The Geiringer-Laman Theorem and Hall's marriage theorem show that it suffices to have that monotonicity alone to guarantee the (weak) Lefschetz property. We arrive at the following illustrative question.

Question 10.3. *Consider a triangulated sphere Σ of even dimension $2k=d-1$, and $\mathscr{A}^*(\Sigma)$ an Artinian reduction such that for every monomial ideal (or any ideal?) $\mathscr{I}$, we have an inequality*

$$\dim \mathscr{I}^k \ \leq\ \dim \mathscr{I}^{k+1}.$$

Does $\mathscr{A}^*(\Sigma)$ have the weak Lefschetz property?

10.3. **Limits of the theory: Cycles.** In [APP23], the authors went a step further: Beyond manifolds, they proved that for a field of characteristic 2, and μ any simplicial homology cycle, $\mathscr{B}(\mu)$ (see Remark 4.11) satisfies total anisotropy, and therefore the Lefschetz property.

Question 10.4. *Can the methods of this paper be extended to cycles?*

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