THE LEFSCHETZ PROPERTY FOR FACE RINGS

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ABSTRACT. We give a simplified version of the Lefschetz property for spheres and homology manifolds in any characteristic.

1. INTRODUCTION

The paper is devoted to proving the following theorem:

Theorem 1.1. *Fix an infinite field $\Bbbk$ of any characteristic. Consider a triangulated compact, closed, $\Bbbk$-orientable $\Bbbk$-homology manifold M of dimension $d-1$, and the associated graded commutative face ring $\Bbbk[M]$. Then the Gorensteinification of the generic Artinian reduction $\mathscr{A}^*(M)$ of $\Bbbk[M]$ satisfies the weak Lefschetz property with respect to a generic $\ell \in \mathscr{A}^1(M)$, that is, for every $k \leq {}^d/_2$, the map*

$$\mathscr{B}^k(M) \xrightarrow{\ \cdot\ell\ } \mathscr{B}^{k+1}(M)$$

is injective.

We call this the **generic weak Lefschetz property**. Here $\mathscr{B}^*$ is the Gorensteinification with respect to the unique socle element of degree d, that is, the minimal quotient of $\mathscr{A}^*$ that does not trivialize the degree d component of the ring. In the case that M is a $\Bbbk$-homology sphere, $\mathscr{A}^*(M) = \mathscr{B}^*(M)$.

Let us make two notes.

- It is well known that it suffices to prove this for manifolds of even dimension $2k$, and for the map
$$\mathscr{B}^k(M) \xrightarrow{\ \cdot\ell\ } \mathscr{B}^{k+1}(M).$$
In this case, the map is an injection if and only if it is an isomorphism, as both spaces are of the same dimension.
- One can extend this rather straightforwardly to the strong (or hard) Lefschetz property. We will do so at the end, as the weak Lefschetz property stated above is enough for almost all combinatorial applications. The hard Lefschetz property is, in the setting of Theorem 1.1, an isomorphism
$$\mathscr{B}^k(M) \xrightarrow{\ \cdot\ell^{d-2k}\ } \mathscr{B}^{d-k}(M).$$

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This is a streamlined version of the author's original proof of the g-conjecture. It uses some simplifications and alternative constructions, sacrifices some generality of the algebraic theorems for simplicity, and works in arbitrary characteristic from the start. It still contains the other corollaries of the Lefschetz property, but otherwise cuts the exposition short.
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Convention 1.2. Henceforth, a manifold is simply a triangulated $\Bbbk$-homology manifold over the chosen field $\Bbbk$, and a sphere is a $\Bbbk$-homology manifold with the $\Bbbk$-homology of a sphere. Furthermore, all simplicial complexes are finite, and hence all manifolds are compact. Homology is defined over the given field, and only sometimes repeated for emphasis. Similarly, we will sometimes emphasize the nature of the manifold if some result *does not* extend to homology manifolds, etc. Orientability will be defined with respect to the chosen field. Similarly, Betti numbers, (co)homology, etc. are always stated with respect to the chosen field. All manifolds of positive dimension are connected.

1.1. **Applications to combinatorics.** We recall some of the consequences of Theorem 1.1 following [Adi18], though in a slightly greater generality from the start.

(1) *McMullen's g-conjecture, see* [McM71]*:* A direct application of Theorem 1.1, restricted to spheres, is the characterization of its face vector, following and resolving the g-conjecture of McMullen.

In particular, it proves that the f-vector, that is, the number of vertices, edges, two-dimensional faces, etc. of a given sphere Σ is also the f-vector of some simplicial polytope (or in other words, the set of f-vectors of spheres coincides with the set of f-vectors of simplicial polytopes). This is achieved by realizing, using the weak Lefschetz property, that the f-vector is but a nonnegative combination of the entries of the g-vector, that is, the numbers

$$g_i(\Sigma) \ = \ \dim \mathscr{A}^i(\Sigma)\big/\ell\mathscr{A}^{i-1}(\Sigma)$$

of $\mathscr{A}^*(\Sigma)$, see also [Sta80], where $\dim = \dim_{\Bbbk}$ (and henceforth, unless otherwise made explicit).

The g_i are therefore the Hilbert vector of a commutative graded algebra generated in degree one (also known as M-vector), and it was proven by Billera and Lee [BL80] that, conversely, every M-vector can be realized as the g-vector of a simplicial polytope. Hence, face vectors of simplicial polytopes and face vectors of spheres coincide. We refer to Stanley's aforementioned work that discusses the g-conjecture, and its relation to the Lefschetz theorem in detail.

Going beyond spheres, the generic Lefschetz property provides necessary conditions on the possible face vectors of triangulations of a fixed manifold M as well.

(2) *Kühnel conjectures:* In particular, the Lefschetz properties also apply to bound the complexity of triangulated (closed) manifolds, and resolve a conjecture of Kühnel [Küh95]: If M is a triangulated $(d-1)$-dimensional closed manifold on n vertices, then the generic Lefschetz property applied to each link implies that in terms of its reduced Betti numbers over $\Bbbk$, that is

$$\mathrm{b}_*(M) = \mathrm{b}_*(M;\Bbbk) = \dim \widetilde{H}_*(M;\Bbbk),$$

we have

$$\binom{d+1}{j}\mathrm{b}_{j-1}(M) \ \le \ \binom{n-d+j-2}{j} \quad \text{for } 1 \le j \le \frac{d}{2}.$$

This extends to manifolds with boundary, in a way. In that case, we have

$$\binom{d}{j}\mathrm{b}_{j-1}(M) \ + \ \binom{d}{j-1}\mathrm{b}_{d-j}(M) \ \le \ \binom{n-d+j-1}{j} \quad \text{for } 1 \le j \le \frac{d}{2},$$

though for non-orientable manifolds, this seemingly requires the Hard Lefschetz property. We provide a simple proof of both implications in Section 5.

(3) *Complexity measures for connectivity:* Given a sphere Σ of dimension $d-1$, and a nonnegative integer k less than d, one can measure the "complexity" of the triangulation in dimension $k-1$ in several ways. Consider for instance $\widetilde{H}_{k-1}(\Sigma_W)$ for W a subset of vertices $\Sigma^{(0)}$ of Σ and Σ_W the induced subcomplex on W. For a numerical measure, one often uses either the 1-norm

$$|(\Sigma)|_{k-1,1,m} \coloneqq \sum_{\substack{W\subset\Sigma^{(0)}\\ |W|=m}} \mathrm{b}_{k-1}(\Sigma_W)$$

or the ∞-norm

$$|(\Sigma)|_{k-1,\infty} \coloneqq \max_{W\subset\Sigma^{(0)}} \mathrm{b}_{k-1}(\Sigma_W).$$

Following [MN03] and using the Lefschetz property, it follows that $|(\Sigma)|_{k-1,1,m}$ is minimized by the Billera-Lee polytopes among those with the same f-vector, see also Codenotti, Santos and Spreer [CSS18].

Moreover, it follows from [Adi17, Section 6] and the Lefschetz property that for $k \le {}^{d}/{}_{2}$, we have a bound

$$|(\Sigma)|_{k-1,\infty} \le\ g_k(\Sigma).$$

Global combinatorial consequences of this are obtained if g_k vanishes, for instance the generalized lower bound theorem of Murai-Nevo [MN13] for general simplicial homology spheres.

In addition, there exists a subset $\mathscr{E} \subset \Sigma^{(0)}$,

$$|\mathscr{E}| \le\ ((k+1)g_{k+1} + (d+1-k)g_k)(\Sigma)$$

such that $\mathrm{b}_{k-1}(\Sigma_{W\cup\mathscr{E}}) = 0$ for all $W \subset \Sigma^{(0)}$. Hence, nontrivial homology classes in $H_{k-1}(\Sigma_W)$ are not only boundaries in Σ, but one can find them to be boundaries of chains with at most

$$((k+1)g_{k+1} + (d+1-k)g_k)(\Sigma)$$

additional vertices.

By adapting [Adi17, Section 6] to manifolds, one can show the two results mentioned previously still apply to measure the complexity of triangulated manifolds M, but now measuring not the induced homology of subcomplexes, but the dimension of the kernel of

$$\widetilde{H}_{k-1}(M_W) \ \longrightarrow\ \widetilde{H}_{k-1}(M).$$

(4) *Grünbaum-Kalai-Sarkaria conjecture, see* [Gr70]*:* A central application is to a conjecture of Grünbaum, Kalai and Sarkaria: The generic Lefschetz property for PL spheres implies that if Δ is a simplicial complex of dimension d that allows a PL embedding into $\mathbb{R}^{2d}$ (or, equivalently, S^{2d}) then

$$f_d(\Delta)\ \le\ (d+2)f_{d-1}(\Delta). \tag{1}$$

We will give a proof of this implication in Section 5.3, and in fact it motivates much of the inductive structure of the proof. This bound is essentially sharp, only differing from constructions in an additive error depending only on d, see also Remark 5.10. This does extend to spheres and manifolds, if we specify what we mean by PL embedding.

The case $d = 1$ (when Δ is a simple graph) is a consequence of what is now called Euler's formula, postulated first by Descartes and Euler [Des21, Eul58], though the first formal proof by modern standards seems to be due to Legendre [Leg94].

However, already for $d = 2$, only a little improvement was known over the trivial bound of $f_2(\Delta) \leq f_0(\Delta) f_1(\Delta)$: Dey [Dey93] proved that

$$f_d(\Delta) \leq C f_0^{d+1-\frac{1}{3^d}}(\Delta),$$

a bound that stood essentially uncontested for 25 years. Parsa [Par18] recently improved this to

$$f_d(\Delta) \leq C f_0^{d+1-\frac{1}{3^{d-1}}}(\Delta)$$

but the topological techniques employed are seemingly limited in potential.

Our Inequality (1) implies the stronger and optimal

$$f_d(\Delta) \leq (d+2) \binom{f_0(\Delta)}{d}.$$

We should remark that these bounds are contingent on the map being PL, and will discuss the somewhat subtle issue of non-PL maps when we discuss the derivation from the Lefschetz property. Beyond the PL case, the best bounds seem to be due to Parsa [Par18] and Patak-Tancer [PT19].

The bound we obtain for PL maps has various consequences in discrete geometry (see also [Wag13, Section 3]). For instance, if Γ is a simplicial complex and $\Gamma \to \mathbb{R}^{2d}$ is a PL map that is locally injective, that is, it is injective on neighborhoods of vertices, then we have for the number of pairwise intersections of d-simplices, the d-th crossing number $\mathrm{cr}_d(\Gamma)$, the inequality

$$\mathrm{cr}_d(\Gamma) \geq \frac{f_d^{d+2}(\Gamma)}{(d+3)^{d+2} f_{d-1}^{d+1}(\Gamma)}$$

if $f_d(\Gamma) > (d+3) f_{d-1}(\Gamma)$, therefore obtaining a generalization of the influential crossing lemma of Ajtai, Chvátal, Leighton, Newborn and Szemerédi [ACNS82, Lei83]. We will discuss more details, and generalizations to manifolds, in Section 5.3.

2. Key ideas and overview

The idea for the proof is this: We show in ***Section 7*** that it is easy to show the Lefschetz property for certain kinds of manifolds which have a nice decomposition. We in particular introduce the weakly deconstructible spheres and balls and show the Lefschetz property for these. ***Section 4*** refines and reworks combinatorial commutative algebra to further the understanding of face rings in terms of simplicial cohomology, and ***Section 6*** uses this to show the Lefschetz property is equivalent to a nondegeneracy property of the Poincaré pairing at subspaces. We call this *the biased pairing property*.

The idea is now that the biased pairing property (which is equivalent to the Lefschetz property) does not have to be proven by proving the Lefschetz property on the original manifold directly, which may be hard to decompose, can instead be proved on a different manifold. ***Section 8*** shows that indeed, we can replace the starting manifold with a simpler one, that has a nice decomposition (and one that is locally weakly deconstructible). ***Section 9*** just puts things together then.

Here is a more verbose overview of the paper.

- ***Section 3*** gives some of the basics for the rings we use and recalls some classical facts.
- ***Section 4*** reworks much of combinatorial commutative algebra in a way that gives us more concrete understanding of the connection between simplicial cohomology and commutative algebra, in addition to obtaining new results and explaining some phenomena between topology and commutative algebra in a more hands-on way.
- ***Section 5*** will recall a basic classical case of a generic Lefschetz theorem, and its relations to an expansive numerical property of ideals in face rings. It provides intuition and motivation for the notion of biased pairings. The section also proves, using the previous sections, some consequences of the Lefschetz property, such as the Grünbaum-Kalai-Sarkaria conjecture (Section 5.3) and a collection of conjectures of Kühnel (Section 5.5) concerning generalizations of the Heawood inequalities.
- ***Section 6*** introduces the *biased pairing property* (Section 6.2). Essentially, we argue that in the setting beyond positivity, where we have no control over signature, it becomes universally important to understand the Poincaré pairing on ideals. The property has several uses throughout. The key in this section is to establish a *characterization theorem for biased pairings*. We start from the basic case that *relates it to the Lefschetz property for spheres* (Proposition 6.6).

  The stronger, most general version, characterizes *the biased pairing property in terms of a Lefschetz type property for general hypersurfaces* (Section 6.8 and Theorem 6.16). Hence, the reader should think of this as "biased pairing property equivalent to Lefschetz property (in codimension one)".
- ***Section 7*** discusses how to obtain Lefschetz properties from biased pairing properties. What we want is to understand the multiplication with a generic element in the face ring. This seems unachievable at first.

  So instead, we take a multiplication in the ring we understand, the one by a variable in the face ring. Now we inductively add more variables in a generic combination to understand the multiplication with a more general linear form. Ideally, *the kernel of multiplication with a generic combination (of variables) is the intersection of the kernels;* this is the property we call *transversality of primes* (Section 7.2). If we prove it, we are done, for the intersection of kernels of all the variables is 0 by Poincaré duality.

  To actually prove the transversal prime property, we employ a *perturbation lemma* (Section 7.1) that goes back at least to Kronecker: to guarantee the generic combination of two linear maps has minimal kernel, we give a simple *transversality criterion that reduces to a biased pairing property itself*. We shall see that it is rather straightforward to prove the transversal prime property if we have a sufficiently nice decomposition of the manifold in question, and illustrate this by introducing the notion of weak deconstructibility, and derive the Lefschetz property in that case. While this class is rather special, we will see that the Lefschetz property for *any sphere or manifold reduces to that of weakly deconstructible spheres and balls.*
- At this point, we diverge significantly from [Adi18]: In that paper, we gave a construction, called **railways** that shows that to prove the biased pairing property (and equivalently by Section 6, the Lefschetz property for a manifold M), one can replace M with a nicely decomposed manifold. ***Section 8*** introduces a simplified railway construction that reduces the Lefschetz property for all spheres to those of weakly deconstructible complexes.

◦ ***Section 9*** then puts things together, making a few additional observations until we formally reduce the Lefschetz property for all spheres to the Lefschetz property for weakly deconstructible spheres.
◦ Finally, we discuss the hard Lefschetz property and some open problems in ***Section 10***.

3. Basic Notions

We set up some of the basic objects, and refer to [AY21, Lee96, Sta96] for a more comprehensive introduction. Especially the first reference follows our viewpoint closely. Starting with Section 4, we investigate the Poincaré pairing and socle. We fix an infinite field $\mathbb{k}$.

3.1. **Face rings.** If Δ is an abstract simplicial complex on groundset $[n] := \{1, \cdots, n\}$, let $I_\Delta := \langle \mathbf{x}^\mathbf{a} : \operatorname{supp}(\mathbf{a}) \notin \Delta \rangle$ denote the nonface ideal in $\mathbb{k}[\mathbf{x}]$, where $\mathbb{k}[\mathbf{x}] = \mathbb{k}[x_1, \cdots, x_n]$. As a general convention, all our simplicial complexes (unless they are relative) have the empty set as a face. This also means that homology and cohomology of simplicial complexes is naturally always reduced.

Let $\mathbb{k}^*[\Delta] := \mathbb{k}[\mathbf{x}]/I_\Delta$ denote the face ring of Δ. A collection of linear forms $\Theta = (\theta_1, \cdots, \theta_l)$ in the polynomial ring $\mathbb{k}[\mathbf{x}]$ is a **partial linear system of parameters** if

$$\dim_{\mathrm{Krull}} \mathbb{k}^*[\Delta] \big/ \Theta \mathbb{k}^*[\Delta] \ = \ \dim_{\mathrm{Krull}} \mathbb{k}^*[\Delta] - l,$$

for $\dim_{\mathrm{Krull}}$ the Krull dimension. If $l = \dim_{\mathrm{Krull}} \mathbb{k}^*[\Delta] = \dim \Delta + 1$, then Θ is simply a **linear system of parameters**, and the corresponding quotient $\mathbb{k}^*[\Delta]/\Theta\mathbb{k}^*[\Delta]$ is called an **Artinian reduction** of $\mathbb{k}^*[\Delta]$.

3.2. **The relative case.** A **relative simplicial complex** $\Psi = (\Delta, \Gamma)$ is a pair of simplicial complexes Δ, Γ with $\Gamma \subset \Delta$. If $\Psi = (\Delta, \Gamma)$ is a relative simplicial complex, then we can define the **relative face module**

$$\mathbb{k}^*[\Psi] \ := \ I_\Gamma \big/ I_\Delta$$

and its reduction $\mathbb{k}^*[\Psi]/\Theta\mathbb{k}^*[\Psi]$.

3.3. **Coordinates and properness.** Observe finally that Θ induces a map $\Delta^{(0)} \to \mathbb{k}^l$ by associating to the vertices of Δ the coordinates $\mathbf{V}_\Delta = (v_1, \cdots, v_n) \in \mathbb{k}^{l \times n}$, where $\mathbf{V}_\Delta \mathbf{x} = \Theta$. Hence, as is canonical when considering toric varieties, we identify a pair $(\Delta; \Theta)$ with a **geometric simplicial complex**, that is, a simplicial complex with a map of the vertices to $\mathbb{k}^l$.

Conversely, the canonical reduced face rings of a geometric simplicial complex are those induced by the linear system of parameters given by the geometric realization. The face ring of a **geometric simplicial complex** is considered with respect to its natural system of parameters induced by the coordinates.

A geometric simplicial complex in $\mathbb{k}^d$ is **proper** if the image of every k-face, with $k < d$, linearly spans a subspace of dimension $k + 1$. A sequence of linear forms is a (partial) linear system of parameters if the associated coordinatization is proper (this is an easy exercise, see for instance [Lee96, Section 4]). For the results of our paper, we always think of every simplicial complex as geometric and proper, that is, as coming with a proper coordinatization in a vector space over $\mathbb{k}$, and shall generally assume $(d-1)$-dimensional complexes to be realized in $\Bbbk^d$ unless otherwise stated, so that the associated collection of coordinatizing linear forms is a linear system of parameters.

We are most interested in the cases when the coordinates give rise to an Artinian reduction, that is, the coordinates induce a linear system of parameters, justifying our notation. All geometric simplicial complexes are therefore assumed to be proper. For geometric simplicial complexes, we shall use the notation

$$\mathscr{A}^*(\Psi) \ \coloneqq\ \Bbbk^*[\Psi; \mathbf{V}_\Psi \mathbf{x}].$$

The (ideal generated by) $\mathbf{V}_\Psi \mathbf{x}$ will therefore also be called the **coordinate ideal of parameters** for a geometric simplicial complex.

3.4. **Conewise polynomials.** It is occasionally useful to remember an alternative model for face rings: the ring of conewise polynomials, see for instance [BBFK02, Bri97]. Consider a simplicial complex Δ as above, together with a proper realization over $\Bbbk$, that is, with vertex coordinates in some vector space V over $\Bbbk$. For every face σ, consider the algebra $\mathscr{P}(\sigma)$ of polynomial functions from the span of σ in V to $\Bbbk$, and denote by $\mathscr{P}(C)$ the algebra of elements in $\bigoplus_{\sigma\in\Delta}\mathscr{P}(\sigma)$ that commutes with restriction to subspaces:

$$(f_\sigma)_{|\mathrm{span}(\tau)} \ =\ f_\tau$$

It is an easy exercise to see that $\Bbbk[\Delta]\ \cong\ \mathscr{P}(C)$. This is achieved by sending an indeterminate x_v corresponding to the vertex v to the conewise linear function that vanishes on the conepoint and on every ray of C except for the one corresponding to v, where it takes value 1 on v itself. As both algebras are generated in degree one, this determines the desired isomorphism. Notice that in the realized case the linear system of parameters has a particularly nice image: it consists of the restrictions of global linear functions.

3.5. **The cone lemmas.** A crucial ingredient for the inductive structure is given by pullbacks to variables corresponding to vertices. Recall that the **star** and **link** of a face σ in Δ are the subcomplexes

$$\mathrm{st}_\sigma\,\Delta \ \coloneqq\ \{\tau : \exists \tau' \supset \tau,\ \sigma\subset\tau'\in\Delta\} \ \text{ and } \ \mathrm{lk}_\sigma\,\Delta \ \coloneqq\ \{\tau\backslash\sigma : \sigma\subset\tau\in\Delta\}.$$

For geometric simplicial complexes Δ, we shall think of the star of a face as a geometric subcomplex of Δ, and the link of a face σ as the geometric simplicial complex obtained by the projection with kernel spanned by σ. Let us denote the **deletion** of σ by $\Delta-\sigma$, the maximal subcomplex of Δ that does not contain σ. Let

$$\mathrm{st}^\circ_\sigma\,\Delta \ \coloneqq\ (\mathrm{st}_\sigma\,\Delta, \mathrm{st}_\sigma\,\Delta-\sigma).$$

We have the following two elementary lemmas.

Lemma 3.1 (Cone lemma I, see [Lee96, Thm. 7] or [TW00])**.** *For any vertex $v\in\Delta$, where Δ is a geometric simplicial complex in $\Bbbk^d$, and any integer k, we have an isomorphism*

$$\mathscr{A}^k(\mathrm{lk}_v\Delta)\ \cong\ \mathscr{A}^k(\mathrm{st}_v\,\Delta).$$

Lemma 3.2 (Cone lemma II, see [Adi17, Lem. 3.3])**.** *In the situation of the first cone lemma we have a natural isomorphism*

$$\cdot x_v : \mathscr{A}^k(\mathrm{st}_v\, \Delta) \longrightarrow \mathscr{A}^{k+1}(\mathrm{st}_v^\circ\, \Delta)$$

where x_v acts by multiplication.

4. Partition of unity and Poincaré duality

The approach to combinatorial commutative algebra introduced as the **partition complex** turns out to be quite fruitful. It in particular gives a new proof of Reisner's theorem (that was claimed by Stanley and Walker in [BGS82]). In particular, all basic facts used in this section are also reproven using the techniques introduced here, see also [AY21] for a more verbose version.

Recall that a simplicial complex Δ of dimension $d-1$ is **Cohen–Macaulay** (over $\mathbb{k}$) if the link of a $k-1$-dimensional face carries reduced $\mathbb{k}$-homology only in dimension $d-1-k$. Following a theorem of Hochster [Sta96, Section II.4], this is equivalent to the following statement: Consider the face ring $\mathbb{k}^*[\Delta]$ and any linear system of parameters Θ. Consider the associated Koszul complex $K^\bullet(\Theta)$.

Then Δ is Cohen-Macaulay if and only if $\mathbb{k}^*[\Delta] \otimes K^\bullet(\Theta)$ carries homology only in final nontrivial component, where the homology is precisely $\mathscr{A}^*(\Delta, \Theta)$.

We can now state:

Lemma 4.1 (Partition of unity)**.** *Consider a Cohen–Macaulay $(d-1)$-complex Δ in $\mathbb{k}^d$. Then, for every $k < d$, we have an injection*

$$\mathscr{A}^k(\Delta) \hookrightarrow \bigoplus_{v\in\Delta^{(0)}} \mathscr{A}^k(\mathrm{st}_v\, \Delta) \tag{2}$$

Recall here that $\cdot^{(i)}$ denotes the set of i-dimensional faces of a simplicial complex. We consider homology with $\mathbb{k}$-coefficients

Proof. Let Θ denote the coordinate ideal of parameters for Δ. For the proof, we consider the Koszul complex $K^\bullet := K^\bullet(\Theta)$ and the chain complex $\widetilde{\mathscr{P}}^\bullet = \widetilde{\mathscr{P}}^\bullet(\Delta)$ defined as

$$0 \longrightarrow \mathbb{k}^*[\Delta] \longrightarrow \bigoplus_{v\in\Delta^{(0)}} \mathbb{k}^*[\mathrm{st}_v\, \Delta] \longrightarrow \cdots \longrightarrow \bigoplus_{F\in\Delta^{(d-1)}} \mathbb{k}^*[\mathrm{st}_F\, \Delta] \longrightarrow 0$$

with a natural choice of maps for the boundary operator: the choice of signs can be intuited easily once we realize that the degree zero component is naturally the Čech complex of Δ covered by the interiors of stars $\mathrm{st}_v \Delta$ [ES52].

With this, we obtain a chain complex as desired. In positive degree, we simply have an inclusion-exclusion of monomials, so the complex is exact. Let $\mathscr{P}^\bullet = \mathscr{P}^\bullet(\Delta)$ denote the complex

$$0 \longrightarrow \mathscr{A}^*(\Delta) \longrightarrow \bigoplus_{v\in\Delta^{(0)}} \mathscr{A}^*(\mathrm{st}_v\, \Delta) \longrightarrow \cdots \longrightarrow \bigoplus_{F\in\Delta^{(d-1)}} \mathscr{A}^*(\mathrm{st}_F\, \Delta) \longrightarrow 0.$$

Consider the double complex $K^\bullet \otimes \widetilde{\mathscr{P}}^\bullet$ and the associated total complex $\mathrm{Tot}^\bullet$, that is, the complex on summands $\mathrm{Tot}^\ell = \bigoplus_{i+j=\ell} K^i \otimes \widetilde{\mathscr{P}}^j$. We compute the homology of the total complex by examining the two filtrations of the double complex.

In the direction of the Koszul complex, the double complex is exact except for the final homology group (which is the Artinian reduction). Using this at all the proper vertices, we obtain that $H^n(\mathrm{Tot}^\bullet)$ is simply $H^n(\mathscr{P}^\bullet)$, or specifically for $n = d$, the kernel of the map (2).

Now, we need another observation: All spaces involved have another grading, the grading according to degree of polynomials. To not clutter notation, we shall specify the degree in words. For the following analysis, it is useful to remember the stratification along this degree: Consider the double complex in direction of $\widetilde{\mathscr{P}}^\bullet$. This is exact in positive degree, so that only the degree 0 component of the homology in degree 0 survives. Keeping in mind that the Koszul complex increases the degree, it follows that

$$H^k(\mathscr{P}^\bullet) \;\cong\; H^k(\mathrm{Tot}^\bullet) \;\cong\; (H^{d-1})^{\binom{d}{k}}(\Delta)(-k),$$

where $(+j)$ denotes a shift in degree by j. Hence, we see that $H^d(\mathscr{P}^\bullet) \cong (H^{d-1}(\Delta))(-d)$, which is nontrivial only in degree d. But the degree k component of $H^d(\mathscr{P}^\bullet)$ is isomorphic to the kernel of the Map (2), as desired. □

An important corollary (and equivalent restatement for spheres) of this fact is Poincaré duality for simplicial homology spheres, [Sta96], proved initially using techniques from commutative algebra; note though that the preceding statement is more general in that it applies beyond the case of spheres to Cohen–Macaulay complexes that do not have a fundamental class.

Theorem 4.2. *Let Σ be a $(d-1)$-dimensional simplicial homology sphere in $\Bbbk^d$. Then $\mathscr{A}^*(\Sigma)$ is a Poincaré duality algebra.*

For this, we first notice:

Lemma 4.3. *In any closed orientable manifold M of dimension $d-1$ in $\Bbbk^d$, we have an injection*

$$\mathscr{A}^*(\mathrm{st}^\circ_v M) \;\hookrightarrow\; \mathscr{A}^*(M)$$

for any vertex v of M.

Proof. As shown by Schenzel, [Sch82, Section 4.1 and Lemma 6.3.4], the kernel of

$$\Bbbk^*[M] \big/ \langle \theta_1, \cdots, \theta_{k-1} \rangle \xrightarrow{\cdot\theta_k} \Bbbk^*[M] \big/ \langle \theta_1, \cdots, \theta_{k-1} \rangle$$

can be understood in terms of cohomology of the underlying manifold M.

We apply this to the exact sequence

$$0 \;\longrightarrow\; \Bbbk^*[\mathrm{st}^\circ_v M] \;\longrightarrow\; \Bbbk^*[M] \;\longrightarrow\; \Bbbk^*[M-v] \;\longrightarrow\; 0,$$

computing the Artinian reduction using the above facts and by exploiting that the inclusion of manifolds

$$M - v \;\hookrightarrow\; M$$

induces a surjection in cohomology

$$H^*(M) \;\twoheadrightarrow\; H^*(M-v). \qquad \square$$

Instead of Schenzel's work, one can also use the partition complex directly, see [AY21].

Proof of Theorem 4.2. It suffices to observe that the annihilator of the irrelevant ideal in $\mathscr{A}^*(\Sigma)$ is concentrated in degree d. But then partition of unity implies for all $k < d$ an injection

$$\mathscr{A}^k(\Sigma) \hookrightarrow \bigoplus_{v\in\Sigma^{(0)}} \mathscr{A}^k(\mathrm{st}_v\,\Sigma) \cong \bigoplus_{v\in\Sigma^{(0)}} \mathscr{A}^{k+1}(\mathrm{st}_v^\circ\,\Sigma) \hookrightarrow \bigoplus_{v\in\Sigma^{(0)}} \mathscr{A}^{k+1}(\Sigma)$$

where the middle map is induced by the second cone lemma for each vertex v of Σ. □

4.1. **Manifolds and socles.** For triangulated spheres, we will be interested mostly in the intersection rings $\mathscr{A}^*$, as the above section shows that this results in Poincaré duality algebras already. For manifolds (over $\Bbbk$), we will have to go beyond that because of our desire to work strongly with the Poincaré pairing in the face ring.

For a simplicial $(d-1)$-manifold M that is realized in $\Bbbk^d$,

$$\mathscr{Soc}(M) \subset \mathscr{A}^*(M)$$

shall denote the annihilator of the irrelevant ideal in $\mathscr{A}^*(M)$, also known as the **socle** of $\mathscr{A}^*(M)$. If M is closed, orientable (over $\Bbbk$) and connected, then $\mathscr{Soc}^d$ is generated by one element, the fundamental class, and it coincides with the fundamental class for simplicial cohomology of M. We shall elaborate on the connection between cohomology and the socle in a few moments.

We will denote by $\mathscr{Soc}^\circ(M)$ the restriction of $\mathscr{Soc}(M)$ to degrees $\le d-1$. We call this the **open socle**. Note that with this definition, we have immediately

Lemma 4.4. *For closed orientable $(d-1)$-manifolds M in $\Bbbk^d$ we have*

$$\mathscr{Soc}(M) \cong H^d(\mathscr{P}^\bullet(M))$$

Proof. This is immediate from Lemma 4.3, as in the proof of Theorem 4.2. □

We have the following useful observation:

Proposition 4.5. *For a closed orientable triangulated $(d-1)$-manifold M in $\Bbbk^d$,*

$$\mathscr{A}^*(M)\big/\mathscr{Soc}^\circ(M)$$

is a Poincaré duality algebra. Equivalently, it satisfies partition of unity as stated above, that is, in degrees $k < d$, the map

$$\mathscr{A}^*(M)\big/\mathscr{Soc}^\circ(M) \longrightarrow \bigoplus_{v\in M^{(0)}} \mathscr{A}^*(\mathrm{st}_v M)$$

is injective.

While this fact on its own is interesting, trying to prove partition of unity for $\mathscr{A}^*(M)$ reveals an even more remarkable fact, as it reveals that the socles can be naturally identified using the spectral sequence coming from the two filtrations of the double complex in the proof of Lemma 4.1 above. Computing the second page of both filtrations (i.e., the total complex), and using the fact that stars of nontrivial faces are Cohen–Macaulay, we obtain:

Proposition 4.6. *For a triangulated $(d-1)$-dimensional manifold M in $\Bbbk^d$, we have an exact sequence*

$$0 \longrightarrow (H^{k-1})^{\binom{d}{k}}(M) \longrightarrow \mathscr{A}^k(M) \longrightarrow \bigoplus_{v\in M^{(0)}} \mathscr{A}^k(\mathrm{st}_v\,M). \tag{3}$$

More generally, the degree $(k-i)$-component of $H^{d-i}(\mathscr{P}^\bullet(M))$ is isomorphic to $(H^{k-1})^{\binom{d}{k-i}}(M)$.

Proof. The proof is virtually identical to the proof of Lemma 4.1: The fact that $\widetilde{\mathscr{P}}^\bullet$ is exact in positive degree is untouched, so the computation in that direction of the double complex is untouched. In the direction of the Koszul complex, notice that the $\Bbbk^*[\mathrm{st}_\sigma M] \times K^\bullet(\Theta)$ has homology concentrated in the last component for every non-empty face σ. Hence, we obtain $H^n(\mathrm{Tot}^\bullet) \cong H^n(\mathscr{P}^\bullet)$ for every $n \leq d$. □

For manifolds M, the kernel of

$$\mathscr{A}^*(M) \longrightarrow \bigoplus_{v \in M^{(0)}} \mathscr{A}^*(\mathrm{st}_v\, M)$$

coincides with the so-called Σ-module of [MNY17]. The isomorphism to this module follows immediately by examining our computation of the double complex, or the fact that their Σ-module is a quotient of $\mathscr{A}^k(M)$ that injects into $\bigoplus_{v \in M^{(0)}} \mathscr{A}^k(\mathrm{st}_v\, M)$, and has codimension $(H^{k-1})^{\binom{d}{k}}(M)$ in the former.

Finally, let us note that all the remarks we discussed above, as well as Proposition 4.6, apply for so-called Buchsbaum complexes. These are those simplicial complexes of dimension $d-1$ in which all links of vertices are Cohen-Macaulay of dimension $d-2$.

4.2. **The combinatorial Ishida complex and the map** hom**.** It is useful to recall the combinatorial Ishida complex of Tay and Whiteley [Ish80], called the "rigidity cochain complex" in their work. For a vector space V over $\Bbbk$ of dimension d, let us denote the j-th exterior power by $\bigwedge^j V$. If σ is an i-tensor, then multiplication with it in the exterior algebra induces a natural map

$$\bigwedge^{j} V \to \bigwedge^{j+i} V.$$

Consider a geometric complex Δ in V, and a non-negative integer $k \leq \dim V$. We then define the combinatorial Ishida (cochain) complex $I^\bullet[k]$ as

$$0 \longleftarrow \bigoplus_{\sigma \in \Delta^{(k-1)}} \bigwedge^{d-k} V \Big/ \ker \sigma \longleftarrow \bigoplus_{\tau \in \Delta^{(k-2)}} \bigwedge^{d-k} V \Big/ \ker \tau \longleftarrow \cdots \longleftarrow 0$$

where we identify σ with its $|\sigma|$-tensor and the coboundary of σ is defined as a signed formal sum over the faces of Δ covering σ :

$$\delta c[\sigma] \ = \sum_{\tau \prec \sigma} \mathrm{sign}[\tau, \sigma] c[\tau];$$

which is naturally linearly extended. Indeed, this is just a simplicial cochain complex with a bit more intricate weights. We first recall a classical result.

Lemma 4.7 ([Oda91])**.** *The top cohomology of this complex is isomorphic to $\mathscr{A}^k(\Delta)$.*

Note that $\bigwedge^{d-k} V \big/ \ker\sigma \cong \Bbbk$ for $(k-1)$-dimensional faces σ, and that $\bigwedge^{d-k} V \big/ \ker\tau \cong \Bbbk^{\binom{[d]}{k}}$ for $(k-2)$-dimensional faces τ. Therefore, if we consider the space $C^{k-1}(\Delta) \big/ \delta C^{k-2}(\Delta) \supset H^{k-1}(\Delta; \Bbbk)$, then, appropriately weighted, its $\binom{[d]}{k}$-th power naturally surjects onto the top cohomology of the Ishida complex, and therefore $\mathscr{A}^k(\Delta)$.

Hence, we obtain a map

$$\hom : H^{k-1}(\Delta)^{\binom{[d]}{k}} \longrightarrow \mathscr{A}^k(\Delta) \tag{4}$$

We now have two maps between face rings and simplicial cohomology, one provided by the Ishida complex and the other by Proposition 4.6 and the partition complex and it is natural to ask how they relate. The answer is simple: as the Čech complex inducing the map to homology in the partition complex is exactly the Čech complex associated to the simplicial cohomology given by the Ishida complex. Hence, the map $\hom$ is exactly the map inducing Isomorphism (3).

As a general rule, we will use

$$\mathscr{B}^*(M) := \mathscr{A}^*(M) \Big/ \ker^\circ[\mathscr{A}^*(M) \to \bigoplus_{v\in M^{(0)}} \mathscr{A}^*(\mathrm{st}_v M)]$$

to denote the "partitionable" reduction of $\mathscr{A}^*(M)$ of a manifold M, where

$$\ker^\circ\left[\mathscr{A}^*(M) \to \bigoplus_{v\in M^{(0)}} \mathscr{A}^*(\mathrm{st}_v M)\right] = (H^d)^\circ(\mathscr{P}^\bullet(M))$$

is the restriction of $H^d(\mathscr{P}^\bullet(M))$, to elements of degree at most $\dim M$.

Remark 4.8. If M is closed and orientable, then $\mathscr{B}^*(M)$ is also the quotient of $\mathscr{A}^*(M)$ by the annihilator under Poincaré pairing to the fundamental class: The unique socle element $\mu \in \mathscr{A}^d(M)$ defines a map $\mu^* : \mathscr{A}^d(M) \longrightarrow \Bbbk$.

Spinning this further, if Δ is any geometric simplicial complex of dimension $d-1$ realized in $\Bbbk^d$, then $\mathscr{A}^d(\Delta) \cong H^{d-1}(\Delta)$. Any homology class μ in $H_{d-1}(\Delta)$ defines a map

$$\mu^* : \mathscr{A}^d(\Delta) \cong H^{d-1}(\Delta) \longrightarrow \Bbbk.$$

We can then look at the quotient of $\mathscr{A}^*(\Delta)$ by the annihilator under Poincaré pairing to $\Bbbk$ via μ^*, and denote it by $\mathscr{B}^*(\mu)$. For closed, orientable manifolds $\mathscr{B}^*(\Delta) = \mathscr{B}^*(\mu)$, where the latter is its fundamental class. This viewpoint is taken in [APP23].

With this notation, we obtain also immediately the following version of Lemma 4.3.

Lemma 4.9. *In any closed orientable manifold M of dimension $d-1$ in $\Bbbk^d$, we have an injection*

$$\mathscr{A}^*(\mathrm{st}_v^\circ M) \hookrightarrow \mathscr{B}^*(M)$$

for any vertex v of M.

Hence, we also obtain the proof of Proposition 4.5.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & (H^d)^\circ(\mathscr{P}^\bullet(M-v)) \arrow[r] & \mathscr{A}^*(M-v) \arrow[r] & \mathscr{B}^*(M-v) \arrow[r] & 0 \\
0 \arrow[r] & (H^d)^\circ(\mathscr{P}^\bullet(M)) \arrow[r] \arrow[u, "\wr"] & \mathscr{A}^*(M) \arrow[r] \arrow[u] & \mathscr{B}^*(M) \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[r] \arrow[u] & \mathscr{A}^*(\mathrm{st}_v^\circ M) \arrow[r] \arrow[u] & \mathscr{A}^*(\mathrm{st}_v^\circ M) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The horizontal sequences are exact by definition. It follows from Proposition 4.6 that the left top vertical map is an isomorphism. Hence, the first two vertical sequences are exact, and by the snake lemma, so is then the third, as desired. □

Note that for closed orientable manifolds, Lemma 4.4 implies that $\mathscr{B}^*(M)$ is the quotient of $\mathscr{A}^*(M)$ by its open socle.

4.3. **Manifolds with boundary, and how to talk about Lefschetz in the boundary case.** The results above extend to manifolds with boundary, with a twist. By replacing Schenzel's results with the appropriate relative version in [AS16] (or, again, [AY21] which subsumes that part of the theory in both cases), we obtain

Corollary 4.10. *In any manifold M of dimension $d-1$ in $\Bbbk^d$ with boundary ∂M, let $M^\circ = (M, \partial M)$. We have an injection*

$$\mathscr{A}^*(\mathrm{st}_v^\circ M^\circ) \hookrightarrow \mathscr{B}^*(M^\circ)$$

for any vertex v of M.

We then obtain, by using Proposition 4.6:

Proposition 4.11. *Given an orientable connected manifold M of dimension $d-1$ (possibly with boundary), there is a perfect pairing*

$$\mathscr{B}^k(M) \times \mathscr{B}^{d-k}(M^\circ) \longrightarrow \mathscr{B}^d(M^\circ) \cong \Bbbk$$

where $M^\circ = (M, \partial M)$.

It is now prudent to say: This is the Poincaré pairing of a manifold with boundary. We can immediately formulate a Lefschetz property for manifolds with boundary like this: It is the map factoring as

$$\begin{tikzcd} \mathscr{B}^k(M^\circ) \arrow[r, "\cdot\ell^{d-2k}"] \arrow[d, "\cdot\ell^{d-2k}"] & \mathscr{B}^{d-k}(M) \\ \mathscr{B}^{d-k}(M^\circ) \arrow[ur] & \end{tikzcd}$$

and specifically demanding that the top map is a Lefschetz isomorphism. However, this is simply the largest isomorphism we can demand, and occasionally there are interesting specializations. Consider the case when M is a submanifold of N of the same dimension. Then

$$\mathscr{B}^*(M \subset N) = \mathscr{B}^*(N) \big/ I_M$$

and its dual, the image of $\mathscr{A}^*(N, \overline{M})$ in $\mathscr{B}^*(N)$. These are equally Poincaré dual and therefore can potentially satisfy a Lefschetz property as well, even if they are in general strict quotients $\mathscr{B}^*(M)$ and $\mathscr{B}^*(M^\circ)$, respectively.

4.4. **$\Bbbk$-homeomorphisms.** A notion that will be useful for us going forward is the notion of a $\Bbbk$-homeomorphism.

Definition 4.12. Let $f : X \longrightarrow Y$ be a continuous map, and let $\Bbbk$ be a field. For each $q \geq 0$, let

$$\mathscr{L}_q^X(U) := H_q(X, X \setminus U; \Bbbk), \qquad \mathscr{L}_q^Y(V) := H_q(Y, Y \setminus V; \Bbbk)$$

denote the local homology sheaves on X and Y.

We say that f is a $\Bbbk$-*homeomorphism* if, for every $q \geq 0$, the natural morphism

$$f^{-1}\mathscr{L}_q^Y \longrightarrow \mathscr{L}_q^X$$

is an isomorphism of sheaves.

Equivalently, f is a local $\Bbbk$-homology equivalence if for every $x \in X$ and every $q \geq 0$, the induced map

$$H_q(Y, Y \setminus \{f(x)\}; \Bbbk) \longrightarrow H_q(X, X \setminus \{x\}; \Bbbk)$$

is an isomorphism.

This is useful in several ways. We note the following:

Proposition 4.13. *Consider $f : M \longrightarrow N$ a $\Bbbk$-homeomorphism of manifolds. Then f induces an isomorphism of socles.*

5. Between combinatorics and algebra

We use this section to go over some applications of Theorem 1.1 and relatives, and also introduce some useful notation and more general perspective for the following chapters.

5.1. **Numerical consequences of the Lefschetz theorem.** Let us consider the following first: Let Σ denote a simplicial sphere and let Δ denote a subcomplex of Σ. We define

$$\mathsf{k}_i(\Sigma, \Delta) \ :=\ \dim \mathscr{K}^i(\Sigma, \Delta),$$

where

$$\mathscr{K}^*(\Sigma, \Delta) \ :=\ \ker[\mathscr{A}^*(\Sigma) \to \mathscr{A}^*(\Delta)].$$

Note that these are just the squarefree monomial ideals generated by the monomials supported outside Δ.

Let us note the following consequence of Theorem 1.1.

Observation 5.1. *Consider Σ a $2k$-dimensional sphere in $\Bbbk^{2k+1}$. If $\mathscr{A}(\Sigma)$ has the weak Lefschetz property, then for every subcomplex Δ of Σ, we have*

$$\mathsf{k}_k(\Sigma, \Delta) \ \leq \mathsf{k}_{k+1}(\Sigma, \Delta). \tag{5}$$

and equivalently

$$\dim \mathscr{A}^k(\Delta) \ \geq\ \dim \mathscr{A}^{k+1}(\Delta)$$

This is a trivial consequence of the commutative square

$$\begin{tikzcd}
\mathscr{A}^k(\Sigma) \arrow[r, "\cdot \ell"] & \mathscr{A}^{k+1}(\Sigma) \\
\mathscr{K}^k(\Sigma, \Delta) \arrow[r, "\cdot \ell"] \arrow[u, hook] & \mathscr{K}^{k+1}(\Sigma, \Delta) \arrow[u, hook]
\end{tikzcd}$$

so that an isomorphism (and in particular injection) of the top map implies an injection of the bottom map.

5.2. **The Geiringer-Laman Theorem.** Let us remind ourselves of the "original" generic Lefschetz theorem, known as Laman's rigidity criterion for planar graphs, because it gives an intuition for what happens here. Recall that a **Laman graph** is a graph G on n vertices and $2n-3$ edges such that every subgraph on x vertices has at most $2x-3$ edges.

Observation 5.2. *Assume G is a graph generically embedded into $\Bbbk^2$. Then it is a Laman graph if and only if for every subgraph H of G,*

$$\dim \mathscr{A}^1(H) \ \geq\ \dim \mathscr{A}^2(H),$$

or equivalently

$$\kappa_1(G,H)\ \leq \kappa_2(G,H).$$

Theorem 5.3 (Geiringer-Laman, [PG27, Lam70])**.** *Consider a Laman graph G. Then a generic realization of G in $\mathbb{R}^2$ with straight edges is infinitesimally rigid, that is, every infinitesimal deformation of G that preserves edge-lengths in the first order is a restriction of a global isometry.*

Let us restate the Geiringer-Laman Theorem, using the work of [Lee96], in another form more familiar to us:

Theorem 5.4. *For any generic embedding of a Laman graph G into $\Bbbk^2$, we have an isomorphism*

$$\mathscr{A}^1(G) \ \xrightarrow{\cdot\ell}\ \mathscr{A}^2(G),$$

where ℓ denotes any generic element of $\mathscr{A}^1(G)$. Conversely, any graph on v vertices and $2v-3$ edges admits such an embedding if and only if it is a Laman graph.

In the latter form, the Geiringer-Laman Theorem looks suspiciously like a Lefschetz theorem. Its proof follows a very nice idea; we call it the "decomposition principle", named after the decomposition theorem for maps between algebraic varieties [BBDG18].

Proof Sketch. Every Laman graph can be shown to be obtained from a single edge by the Henneberg moves. One connects two distinct vertices of a graph, the other connects three vertices, two of which are joined by an edge, and deletes the same edge. From the algebraic viewpoint, it is useful to place that new vertex in the second case in the interior of the edge to be deleted. These moves preserve the "Lefschetz property", as seen by an appropriate form of the decomposition theorem: one constructs an embedding of algebras

$$\mathscr{A}^*(G) \ \hookrightarrow\ \mathscr{A}^*(G')$$

encoding the Henneberg move of G to G', gives an appropriate basis for the cokernel and proves the Lefschetz theorem on the decomposition, and in particular concludes rigidity for G' from rigidity for G.

Let us briefly note that there is an easy way to visualize such embeddings of algebras without getting too technical: Using the ring of conewise polynomials (Section 3.4). The embedding is then canonical: we use only the fact that conewise polynomials on $\mathscr{A}^*(G)$ define conewise polynomials on the refinement $\mathscr{A}^*(G')$. □

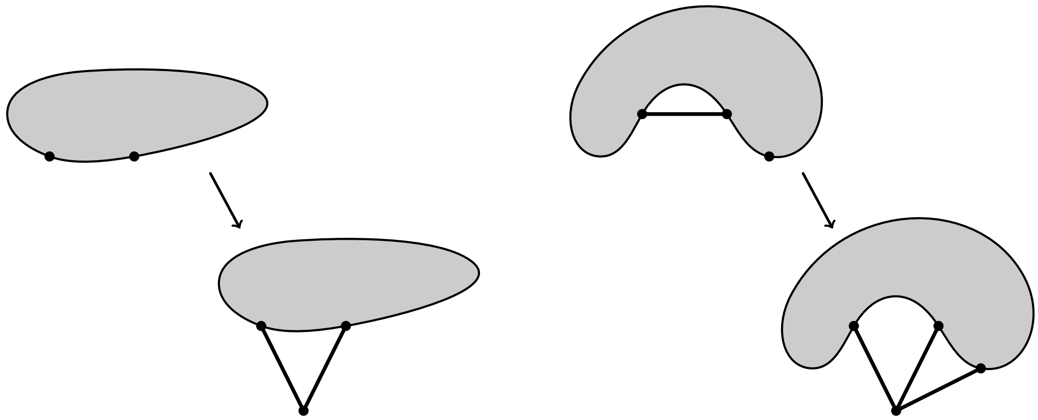

Figure 5.1. The two Henneberg moves.

Hence, in the case of the Geiringer-Laman theorem, monotonicity of ideals (the numbers $\kappa_i(G, H)$) implies a Lefschetz type statement. The paper [Adi18], as well as [PP20, APP23] and the present paper, are devoted to first introducing, and then exploiting a trick to emulate this in higher dimension. Spoiler (for papers that have been out for several years): We do not know yet whether a numerical monotonicity is enough. See also Question 10.3.

5.3. **The Grünbaum-Kalai-Sarkaria conjecture.** We now note a simple but important corollary of Theorem 1.1: The Grünbaum-Kalai-Sarkaria conjecture for sufficiently tame maps. We start with the case of spheres:

Corollary 5.5. *If Σ is a $2k$-dimensional sphere in $\Bbbk^{2k+1}$ that satisfies the weak Lefschetz property, then, for every subcomplex Δ of Σ, we have Inequality (1), that is,*

$$(k+2)\cdot f_{k-1}(\Delta)\ \geq\ f_k(\Delta).$$

Proof. Observation 5.1 gives

$$\dim \mathscr{A}^k(\Delta)\ \geq\ \dim \mathscr{A}^{k+1}(\Delta).$$

Moreover, we have the immediate inequality

$$\dim \mathscr{A}^k(\Delta)\ \leq\ f_{k-1}(\Delta),$$

which follows by estimating the dimension of a vector space by the size of a generating set, in this case the $(k-1)$-faces and

$$\dim \mathscr{A}^{k+1}(\Delta)\ \geq\ f_k(\Delta)-(k+1)f_{k-1}(\Delta),$$

which follows by estimating the number of relations from above: By Oda's Lemma 4.7, there are $k+1$ relations for each $(k-1)$-face, so $(k+1)f_{k-1}(\Delta)$ is an upper bound for the number of nontrivial relations. $\square$

We obtain the Grünbaum-Kalai-Sarkaria conjecture:

Corollary 5.6. *If Δ is a simplicial complex that PL embeds into $\mathbb{R}^{2d}$, then*

$$f_d(\Delta) \leq (d+2)f_{d-1}(\Delta)$$

This follows from Corollary 5.5 and the following folklore fact, see [AP25]:

Lemma 5.7. *Assume Δ is a simplicial complex of dimension d that allows a PL embedding into a manifold M. Then there is a PL triangulation of M that contains Δ as a subcomplex.*

However, not every topological embedding can be deformed to a PL one [DV09, Rus69]. That said, the results extend to appropriate notions of PL embeddings into homology spheres and homology manifolds, which illuminates the notion of "sufficiently tame".

Definition 5.8. We say a simplicial complex X $\Bbbk$**-PL-embeds** into a simplicial complex Y if there is X' a $\Bbbk$-PL refinement of X, and Y' a complex $\Bbbk$-PL equivalent to Y, so that X' is a subcomplex of Y'.

For this, we follow the definition of [AP25]: The neighborhood $N_\sigma X$ of a face σ in a simplicial complex X is the collection of simplices containing it: An **edge contraction** is the operation of removing the neighborhood $N_e X$ of the edge $e = \{0,1\}$ and identifying $N_0 \partial N_e X$ and $N_1 \partial N_e X$. We say that the edge e is **valid** if the contraction is a simplicial complex and call it a **valid edge contraction**.

Examples of $\Bbbk$-PL embeddings include embeddings as subcomplexes, and PL embeddings. We then obtain the following consequence of Theorem 1.1.

Corollary 5.9. *Consider a simplicial complex Δ which is $\Bbbk$-PL embeddable into a closed manifold of dimension $2d$. Then*

$$f_d(\Delta) \ \leq \ (d+2)f_{d-1}(\Delta) + \binom{2d+1}{d}\mathrm{b}_d(M).$$

Proof. Note that it suffices to prove this in the orientable case. In a slight variation of Observation 5.1 and using Proposition 4.5, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{A}^d(M) \arrow[r, "\cdot\ell"] \arrow[d, two heads] & \mathscr{B}^{d+1}(M) \arrow[d, two heads] \\
\mathscr{A}^d(\Delta) \arrow[r, "\cdot\ell"] & \mathscr{A}^{d+1}(\Delta) \big/ \hom H^d(M)^{\binom{2d+1}{d}}
\end{tikzcd}$$

where the top map is a surjection by Theorem 1.1. □

Remark 5.10. Kalai showed that under the assumptions of Corollary 5.5, the algebraic shifting of Δ does not contain the Flores complex $\binom{[2k+3]}{\leq d+1}$ as a subcomplex, and is contained in turn in the shifting of the infinite cyclic $(2k+1)$-polytope [Kal91, Kal01]. This allows one to prove Kruskal-Katona type theorems for complexes embeddable in spheres, and in particular slightly sharper bounds for the Grünbaum-Kalai-Sarkaria conjecture, the equality cases for which are attained by compressed collections of k-faces in cyclic $(2k+1)$-polytopes. In the end, this effort only improves the bounds a little, by an additive error depending on the dimension, so that we shall content ourselves with stating the simplified Bound (1) here.

Remark 5.11 (Kühnel's Heawood type inequalities)**.** The inequality

$$\binom{2d+1}{d}\mathrm{b}_d(M) \geq \dim \mathscr{A}^{d+1}(\Delta) - \dim \mathscr{A}^{d}(\Delta)$$

from the proof of Corollary 5.9 has another consequence towards a conjecture of Kühnel [Küh94]: if a complete k-dimensional complex on n vertices embeds into a closed manifold M, then

$$\binom{n-k-2}{k+1} \ \leq \ \binom{2k+1}{k}\mathrm{b}_k(M),$$

thereby bounding the number of vertices such a complex can have.

5.4. **Beyond closed manifolds.** We already touched upon generalizations to manifolds that are not orientable or closed in the previous section, but let us take a moment to actually formulate the main theorem, and their consequences for generalizing the previous section.

Theorem 5.12. *Fix an infinite field $\mathbb{k}$ of any characteristic. Consider a triangulated orientable $\mathbb{k}$-homology manifold M of dimension $d-1$, and the associated graded commutative face ring $\mathbb{k}[M]$. Then any generic Artinian reduction of $\mathbb{k}[M]$ satisfies the weak Lefschetz property with respect to a generic $\ell \in \mathscr{A}^1(M)$, that is, for every $k \geq \lfloor d/2 \rfloor$, the map*

$$\mathscr{B}^k(M) \ \xrightarrow{\cdot \ell}\ \mathscr{B}^{k+1}(M)$$

is surjective and if $d-1$ is even, then we have an isomorphism

$$\mathscr{B}^{(d-1)/2}(M,\partial M) \ \xrightarrow{\cdot \ell}\ \mathscr{B}^{(d+1)/2}(M)$$

Let us remark that similar hard Lefschetz properties exist. In the setting of this theorem, this is, for $k \leq d/2$, an isomorphism

$$\mathscr{B}^k(M,\partial M) \ \xrightarrow{\cdot \ell^{d-2k}}\ \mathscr{B}^{d-k}(M)$$

and in particular a surjection

$$\mathscr{B}^k(M) \ \xrightarrow{\cdot \ell^{d-2k}}\ \mathscr{B}^{d-k}(M).$$

We address and prove this property in Section 10.

We obtain the following generalizations of the previous section:

Corollary 5.13. *Consider a simplicial complex Δ which is $\mathbb{k}$-PL embeddable into an orientable manifold of dimension $2d$,*

$$f_d(\Delta) \ \leq \ (d+2)f_{d-1}(\Delta) + \binom{2d+1}{d}\mathrm{b}_d(M).$$

Moreover, if a complete k-dimensional complex on n vertices embeds into M, then

$$\binom{n-k-2}{k+1} \ \leq \ \binom{2k+1}{k+1}\mathrm{b}_k(M).$$

5.5. **Some conjectures of Kühnel.** Sometimes, the hard Lefschetz property is useful in combinatorial contexts as well. We now provide a simple way to conclude some conjectures of Kühnel [Küh95] from the hard Lefschetz property (Theorem 10.2) to illustrate the usefulness of the partition complex. In the special case of closed orientable manifolds, this was proven (conditionally on assuming Lefschetz properties) by Murai [Mur15] using a different approach.

Theorem 5.14. *Consider M a triangulated manifold of dimension $d-1$ and on n vertices. Then*

$$\binom{d}{j}\mathrm{b}_{j-1}(M) \;+\; \binom{d}{j-1}\mathrm{b}_{d-j}(M) \;\leq\; \binom{n-d+j-1}{j} \quad \text{for } 1 \leq j \leq \frac{d}{2}.$$

If moreover M is closed, we have the stronger bound

$$\binom{d+1}{j}\mathrm{b}_{j-1}(M) \;\leq\; \binom{n-d+j-2}{j} \quad \text{for } 1 \leq j \leq \frac{d}{2},$$

and

$$\binom{d}{j}\mathrm{b}_{j-1}(M) \;\leq\; \binom{n-d+j-2}{j} \quad \text{for } 1 \leq j \leq d.$$

We note that the proof can also be completed using the weak Lefschetz property using the theory of biased pairings. We will do so in the next Section 6.

Proof. Let us first observe that by Macaulay's theorem [Mac27], it suffices entirely to prove that $\mathscr{A}^j(M)$ is of dimension at least

$$\binom{d}{j}\mathrm{b}_{j-1}(M) \;+\; \binom{d}{j-1}\mathrm{b}_{d-j}(M)$$

Consider now a generic element ℓ of $\mathscr{A}^1(M)$. Consider the diagram

$$\begin{tikzcd}
\mathscr{A}^j(M) \arrow[r, "\cdot\ell^{d-2j+1}"] & \mathscr{A}^{d-j+1}(M) \\
\bigoplus_{v\in M^{(0)}} \mathscr{A}^{j-1}(\mathrm{st}_v M) \arrow[u, two heads] \arrow[r, two heads, "\cdot\ell^{d-2j+1}"] & \bigoplus_{v\in M^{(0)}} \mathscr{A}^{d-j}(\mathrm{st}_v M) \arrow[u, two heads]
\end{tikzcd}$$

where the vertical maps are the cone lemmas, given by the composition

$$\mathscr{A}^{j-1}(\mathrm{st}_v M) \;\simeq\; \mathscr{A}^{j}(\mathrm{st}_v^\circ M) \;\longrightarrow\; \mathscr{A}^{j}(M)$$

where the last map is the inclusion of ideals. We notice that the bottom map is a surjection if $\mathscr{A}(\mathrm{st}_v M)$ has the Lefschetz property for disks. Hence, the top horizontal map is a surjection. By Proposition 4.6, the kernel of the top map is of dimension at least $\binom{d}{j}\mathrm{b}_{j-1}(M)$, and the image is of dimension at least $\binom{d}{j-1}\mathrm{b}_{d-j}(M)$.

Now, if M is even closed, we can go a step further: given the weak Lefschetz property in every link with respect to a generic ℓ, we can extend the Koszul complex by the generic Lefschetz element in the proof of Proposition 4.6 and obtain a short exact sequence

$$0 \;\longrightarrow\; (H^{k-1})^{\binom{d+1}{k}}(M) \;\longrightarrow\; \mathscr{A}^k(M)\big/\ell\mathscr{A}^{k-1}(M) \;\longrightarrow\; \bigoplus_{v\in M^{(0)}} \mathscr{A}^k(\mathrm{st}_v\, M)\big/\ell\mathscr{A}^{k-1}(\mathrm{st}_v\, M)$$

for every $k < d/2$ and for $k \leq d/2$, we have just

$$0 \;\longrightarrow\; (H^{k-1})^{\binom{d+1}{k}}(M) \;\longrightarrow\; \mathscr{A}^k(M)\big/\ell\mathscr{A}^{k-1}(M).$$

For arbitrary j, this additional partitioning property of coprimitive classes fails, so we only have the usual partitioning. These give the desired bound. □

Remark 5.15. There are all kinds of other inequalities one can get from these kinds of arguments. For instance, we have, for a triangulated manifold M of dimension $d-1$ and on n vertices, that

$$\binom{d}{j}\mathrm{b}_{j-1}(M) \ + \ \binom{d}{j'-1}\mathrm{b}_{d-j'}(M) \ \le \ \binom{n-d+j-1}{j} \quad \text{for } 1\le j'\le j\le \frac{d}{2}.$$

5.6. **Partitioned quotients.** Before we continue, it is useful to extend the definition of the modules $\mathscr{K}$ to manifolds M. For subcomplexes Δ of M, we write

$$\mathsf{\kappa}_i(M,\Delta) \ :=\ \dim \mathscr{K}^i(M,\Delta),$$

where

$$\mathscr{K}^*(M,\Delta) \ :=\ \ker[\mathscr{B}^*(M)\to\mathscr{B}^*(\Delta\subset M)]$$

and

$$\mathscr{B}^*(\Delta\subset M) \ :=\ {\mathscr{A}^*(\Delta)}\Big/{\ker\left[\mathscr{A}^*(M)\ \to\ \bigoplus_{v\in M^{(0)}}\mathscr{A}^*(\mathrm{st}_v M)\right]}.$$

It is a useful exercise to understand the dual of $\mathscr{K}^*(M,\Delta)$ under the Poincaré pairing: It is $\mathscr{B}^*(M)_{|\overline{\Delta}}$, the image of

$$\mathscr{B}^*(M)\ \longrightarrow \bigoplus_{\sigma\in M,\ \sigma\notin\Delta}\mathscr{A}^*(\mathrm{st}_\sigma M).$$

If M is closed, orientable and connected, this is precisely the Poincaré dual of $\mathscr{K}^*(M,\Delta)$, see Corollary 6.2. If M has boundary, the same applies with respect to the Poincaré duality for manifolds with boundary.

6. Biased Poincaré duality and the HGL relations: From Lefschetz to pairing properties and back

In this section, we introduce a crucial ingredient of the program by introducing a nondegeneracy property of the Poincaré pairing. This is the crucial ingredient to the proofs of the generic Lefschetz property, as they reduce the Lefschetz property to a pairing problem. This section focuses on the opposite direction first: the implication that the Lefschetz property has towards the biased pairing property.

6.1. **Poincaré pairings.** Recall: Let M be a triangulated (orientable closed connected) manifold of dimension $d-1$ realized in $\mathbb{k}^d$, and $\mathscr{B}^*(M)$ its face ring modulo Artinian reduction and open socle. Then we have a pairing

$$\mathscr{B}^k(M)\ \times\ \mathscr{B}^{d-k}(M)\ \longrightarrow\ \mathscr{B}^d(M)\ \cong\ \mathbb{k}.$$

The essential observation we wish to make is that, while the signature of the pairing does not change under small perturbations, it is often convenient to have the possibility to make certain subspaces non-degenerate under the pairing.

6.2. **Biased Poincaré duality in ideals.** We say that M satisfies **biased Poincaré duality** in degree $k\le\frac{d}{2}$ if for all proper subcomplexes Δ of M, the pairing

$$\mathscr{K}^k(M,\Delta)\ \times\ \mathscr{K}^{d-k}(M,\Delta)\ \longrightarrow\ \mathscr{K}^d(M,\Delta)\ \cong\ \mathbb{k} \tag{6}$$

is nondegenerate on the first factor. Sometimes we will also say that a specific ideal $\mathscr{I}$ in a graded Poincaré duality algebra over $\Bbbk$ satisfies **biased Poincaré duality**, if

$$\mathscr{I}^k \times \mathscr{I}^{d-k} \longrightarrow \mathscr{I}^d \cong \Bbbk \tag{7}$$

is nondegenerate in the first factor, where d is the degree of the fundamental class. An important special case is biased Poincaré duality at a subcomplex Δ, synonymous with biased Poincaré duality for the ideal $\mathscr{K}^*(M, \Delta)$.

We keep it separate as on its own it already has interesting consequences, even without having to consider a Lefschetz theorem. The following is but a restatement of the definition.

Proposition 6.1. *For an ideal $\mathscr{I}$ in $\mathscr{B}^*(M)$ the following are equivalent:*

(1) The map

$$\mathscr{I} \longrightarrow \mathscr{B}^*(M) \Big/ \mathrm{ann}_{\mathscr{B}^*(M)}\mathscr{I}$$

is an injection in degree k.

(2) For every $x \in \mathscr{I}^k$, there exists a y in $\mathscr{I}^{d-k}$ such that $x \cdot y \neq 0$.

(3) $\mathscr{I}$ satisfies biased Poincaré duality in degree k.

We obtain immediately an instrumental way to prove biased Poincaré duality for monomial ideals.

Corollary 6.2. *$\mathscr{K}^*(M, \Delta)$ satisfies biased Poincaré duality in degree k if and only if*

$$\mathscr{K}^k(M, \Delta) \longrightarrow \mathscr{B}^k(M)_{|\overline{\Delta}}$$

is injective.

Proof. It is useful to understand what $\mathscr{B}^*(M)_{|\overline{\Delta}}$ is: following our definition, dualized, it is the image of

$$\mathscr{B}^*(M) \longrightarrow \bigoplus_{\sigma \in M,\ \sigma \notin \Delta} \mathscr{A}^*(\mathrm{st}_\sigma M).$$

Equivalently, it is the quotient of $\mathscr{B}^*(M)$ under the kernel of this map. Which is to say, it is the quotient of $\mathscr{B}^*(M)$ by the annihilator of $\mathscr{K}^k(M, \Delta)$ (as it is the ideal generated by the monomials not supported in Δ). Hence, the corollary follows from the previous proposition. □

6.3. **Relation to the Grünbaum-Kalai-Sarkaria conjecture: Dimension reduction.** Assume that M is a manifold of dimension $2k$ realized in $\Bbbk^{2k+1}$ so that biased Poincaré duality holds with respect to a subcomplex Δ. Then Inequality (5) follows, that is, we have

$$\kappa_k(M, \Delta) \ \leq\ \kappa_{k+1}(M, \Delta).$$

In particular, the Grünbaum-Kalai-Sarkaria conjecture holds for Δ (with the appropriate correction term as in Remark 5.10).

6.4. **General properties of biased Poincaré duality.** To describe and prove biased Poincaré duality in general, we start with a first observation, the following persistence lemma.

Lemma 6.3. *Let M denote a closed orientable $(d-1)$-manifold in $\Bbbk^d$, and $k < \frac{d}{2}$. Then biased Poincaré duality holds for M in degree k if biased Poincaré duality holds for links of all vertices in M and in degree k.*

Proof. By partition of unity in M, any element of $\mathscr{K}^k(M,\Delta)$ pairs with some x_v. By Lemma 4.9, this pullback maps to $\mathscr{A}^k(\mathrm{lk}_v\, M) \cong x_v\mathscr{B}^k(M)$. Using the obvious fact that monomial ideals pull back to monomial ideals, we are reduced to proving Poincaré duality of $\mathrm{lk}_v\,\Delta$ in $\mathscr{A}^k(\mathrm{lk}_v\, M)$; the claim follows. □

This allows us to prove biased Poincaré duality by restricting to the case of degree k and $d = 2k$.

The biased pairing property is rather stable under combinatorial modifications. In the following lemma we focus on stability under subdivisions.

Lemma 6.4. *A PL homeomorphism $\varphi : M \to M'$ of manifolds $M,\ M'$ in $\Bbbk^d$ that restricts to the identity on a common subcomplex Δ preserves biased Poincaré duality at Δ, that is, $\mathscr{K}^*(M,\Delta)$ satisfies biased Poincaré duality (in degree k) if and only if $\mathscr{K}^*(M',\Delta)$ does (in degree k).*

Proof. By the persistence lemma, we may assume that M is of dimension $2k-1$. The idea is to construct an isomorphism
$$\overline{\mathscr{K}}^*(M,\Delta)\ \cong\ \overline{\mathscr{K}}^*(M',\Delta)$$
where $\overline{\mathscr{K}}^*(M,\Delta)$ resp. $\overline{\mathscr{K}}^*(M',\Delta)$ are the orthogonal complements of $\mathscr{K}^*(M,\Delta)$ resp. $\mathscr{K}^*(M',\Delta)$ under the Poincaré pairing. If we do so, we are done, as proving biased Poincaré duality can be proven on $\mathscr{K}^k(M,\Delta)$ or its orthogonal complement under the Poincaré pairing in degree k.

To this end, notice that we may define a PL homeomorphism more generally as any transformation obtained by stellar subdivisions and their inverses. Following Alexander's classical work (see [Lic99] for a survey), this subsumes classical PL equivalences. Recall that a refinement of geometric simplicial complexes X to a complex X' induces a map
$$\mathscr{A}^*(X)\ \longrightarrow\ \mathscr{A}^*(X').$$
This is easily seen on the level of conewise polynomial functions, see Section 3.4. After all, a conewise polynomial on a simplicial cone complex is conewise polynomial on any refinement.

Now, a stellar subdivision of M at a face σ, denoted by $M{\uparrow}\sigma$, induces a pullback map
$$\mathscr{A}^*(M)\ \hookrightarrow\ \mathscr{A}^*(M{\uparrow}\sigma). \tag{8}$$
Its cokernel is a subspace of the ideal of x_{v_σ}: after all, it is generated by those functions that are conewise polynomial on $M{\uparrow}\sigma$, but not on M. But this means we can generate those functions by those that vanish on $M-\sigma$. Which are generated by x_{v_σ}.

Now, the embedding (8) above descends an embedding
$$\mathscr{B}^*(M)\ \hookrightarrow\ \mathscr{B}^*(M{\uparrow}\sigma).$$
This is simply because the kernels of $\mathscr{A}^*(M)\to\mathscr{B}^*(M)$ resp. $\mathscr{A}^*(M{\uparrow}\sigma)\to\mathscr{B}^*(M{\uparrow}\sigma)$ are identified by Proposition 4.6 as homology groups of M, which stay invariant in the subdivision. The cokernel is therefore[1], once again, lying in the ideal of x_{v_σ}. Hence, the cokernel is identical to the cokernel of
$$\mathscr{K}^*(M,\Delta)\ \hookrightarrow\ \mathscr{K}^*(M{\uparrow}\sigma,\Delta).$$

[1] which can be shown to have not changed, though that is irrelevant for us

Hence, $\overline{\mathscr{K}}^*(M, \Delta) \ \cong\ \overline{\mathscr{K}}^*(M{\uparrow}\sigma, \Delta)$ stays invariant under the subdivision, which implies in particular that the non-degeneracy of the Poincaré pairing is preserved. □

Remark 6.5. In fact, the argument above gives that

$$\mathscr{B}^*(M{\uparrow}\sigma) \ \cong\ \mathscr{B}^*(M) \oplus \bigoplus_{i=1}^{|\sigma|-1} x_{v_\sigma}^i \mathscr{A}^*(\mathrm{lk}_\sigma M).$$

Moreover, this decomposition is canonically orthogonal under the Poincaré pairing, see also [Pet16, Proposition 2.2]. For toric varieties, this is but a special case of the decomposition theorem (see [BBDG18]) and the proof in this generality is a straightforward adaption.

6.5. **Characterization theorem for biased Poincaré duality: spherical envelopes.** We now provide the key result that helps us on the way to prove the Lefschetz theorem for spheres. The idea is to show equivalence with another property. Recall that a hypersurface is an embedded manifold of codimension one.

Proposition 6.6. *Assume E is a hypersurface sphere in a closed orientable manifold M of dimension $2k-1=d-1$. Then M in $\Bbbk^d$ satisfies biased Poincaré duality in degree k and with respect to E if and only if*

$$\mathscr{A}^k(E) \ =\ 0.$$

Observation 6.7. *Note that $\mathscr{A}^k(E) \ =\ 0$ is a Lefschetz property: Project E to a hyperplane along a projection map π, and let ϑ denote the height of the vertices over that projection. Then $\mathscr{A}^k(E) \ =\ 0$ if and only if ϑ induces the middle Lefschetz isomorphism on $\mathscr{A}^*(\pi E)$. These cases, where forgetting an element of the linear system that acts as an "additional" linear element in the subcomplex, seem to be, and in the cases of this paper always are, Lefschetz properties.*

The proof is very simple, but illustrates a few important principles.

Proof. We have a short exact sequence

$$0 \ \longrightarrow\ \mathscr{K}^k(M,E) \ \longrightarrow\ \mathscr{B}^k(M) \ \longrightarrow\ \mathscr{A}^k(E) \ \longrightarrow\ 0$$

so we see that

$$\mathscr{B}^k(M) \ =\ \mathscr{K}^k(M,E) \ =\ \mathscr{B}^k(M)_{|\overline{E}}$$

in case the first space is trivial, so the claim simply follows by Poincaré duality.

For the converse, we consider the exact sequence

$$0 \ \longrightarrow\ \mathscr{K}^k(M,E) \ \longrightarrow\ \mathscr{B}^k(M) \ \longrightarrow\ \mathscr{B}^k(E \subset M) \ \longrightarrow\ 0.$$

Notice that because

$$H^{k-1}(M) \ \longrightarrow\ H^{k-1}(E)$$

is the trivial map, we have

$$\mathscr{B}^k(E \subset M) \ \cong\ \mathscr{A}^k(E)$$

by Proposition 4.6. Hence, a nontrivial $\alpha \in \mathscr{A}^k(M)$ that maps nontrivially to $\mathscr{A}^k(E)$ is in the cokernel of the map from the cohomology of E. Consider then the compactification $\widetilde{M}$ of $M \setminus E$ as a manifold with boundary, creating two boundary components isomorphic to E. As α is in the cokernel from the cohomology of E, it is in the cokernel under the cohomology $\widetilde{M}$ (using the inclusion of E into the latter). Hence, α, in either of the two copies of E, has nontrivial image in $\mathscr{B}^k(E \subset \widetilde{M})$ by Proposition 4.6. Under the simplicial map identifying the two boundary components of $\widetilde{M}$ again, the manifold $\widetilde{M}$ gets mapped to M, and $\mathscr{B}^k(M)_{|\overline{E}}$ gets mapped to $\mathscr{B}^k(\widetilde{M})$:

Notice that

$$\mathscr{A}^k(\mathrm{st}_\sigma \widetilde{M}) \ = \ 0$$

unless σ is an interior $(k-1)$-face of $\widetilde{M}$, so that we have a well-defined map

$$\bigoplus_{\substack{\sigma \in M, \\ \sigma \notin E}} \mathscr{A}^k(\mathrm{st}_\sigma M) \ \longrightarrow \ \bigoplus_{\sigma \in \widetilde{M}} \mathscr{A}^k(\mathrm{st}_\sigma \widetilde{M})$$

and in particular a map of the images of these two direct sums in $\mathscr{A}^k(\widetilde{M})$ resp. $\mathscr{A}^k(M)$, i.e. an embedding map

$$\mathscr{B}^k(M)_{|\overline{E}} \ \hookrightarrow \ \mathscr{B}^k(\widetilde{M}).$$

so that then α is nontrivial $\mathscr{B}^k(M)_{|\overline{E}}$.

Hence

$$\mathscr{K}^k(M,E) \ \longrightarrow \ \mathscr{B}_k(M)^{|\overline{E}}$$

has a nontrivial cokernel (as $\alpha = 0$ is nontrivial in the latter space). Because $\mathscr{B}^k(M)_{|\overline{E}}$ and $\mathscr{K}^k(M,E)$ are isomorphic as vector spaces (they are dual under the Poincaré pairing) the map also has a nontrivial cokernel, which implies that M does not satisfy biased Poincaré duality with respect to Δ by Corollary 6.2. □

Example 6.8. Consider the case $k = 1$, and Σ a sphere of dimension 1, realized in $\mathbb{R}^2$. If Δ is a 0-dimensional sphere in Σ, then $\mathscr{K}(\Sigma, \Delta)$ satisfies biased Poincaré duality if and only if Δ does not lie on a line through the origin in $\mathbb{R}^2$.

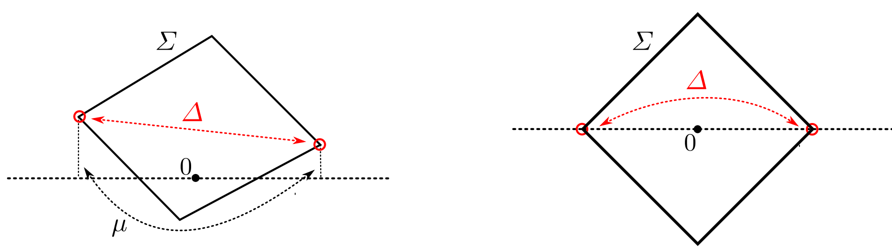

Figure 6.2. Biased Poincaré duality in a sphere Σ and with respect to a codimension one sphere Δ is related to the Lefschetz theorem on the latter.

On the other hand, this means that very nice and even smooth varieties do not satisfy biased Poincaré duality, for instance products of lower-dimensional varieties like the product of two projective lines $\mathbb{P}^1 \times \mathbb{P}^1$, shown in Figure 6.2 on the right.

6.6. **From spheres to complexes to manifolds.** The goal of the remainder of this section is to provide a comprehensive criterion for Lefschetz establishing biased pairing properties for ideals of the form $\mathscr{K}^*(M, \Delta)$. As discussed in [Adi18], this can be reduced to the case when Δ is a special hypersurface, and we will do so in greater generality later, in Section 8. But for this illustrative section, which already presents the key ideas, we work with PL manifolds instead of the more complicated constructions in later in this paper.

Say we want to prove the middle Lefschetz property for a sphere Σ of dimension $2k-2$. By the above Proposition 6.6, we can embed Σ into a manifold M as a hypersurface, and prove a pairing property instead. Clearly, it does not matter if we prove that pairing property for Σ, or its $(k-1)$-skeleton. This provides us with critical flexibility. Indeed, every complex of dimension $(k-1)$ in a combinatorial $(2k-1)$-manifold embeds into many hypersurfaces.

This is based on a folklore geometric observation:

Proposition 6.9. *If Δ is a $(k-1)$-dimensional subcomplex in a $(2k-1)$-dimensional PL manifold M in $\Bbbk^{2k}$, then there exists a subdivision M' of M, and a $2(k-1)$-dimensional hypersurface S in M' containing Δ.*

A notion of importance in this context is the **simplicial neighbourhood** $\mathrm{N}_\Gamma \Delta$ of an induced subcomplex Γ in a simplicial complex Δ, defined as

$$\mathrm{N}_\Gamma \Delta \coloneqq \bigcup_{w \in \Gamma^{(0)}} \mathrm{st}_w \, \Delta.$$

We call a simplicial neighbourhood **regular** if it is regular in the sense of PL topology [RS82], that is, it is a PL mapping cylinder over the PL manifold given as the boundary of the simplicial neighbourhood, mapped to Γ. We sometimes use the notion of regular neighbourhood without referring to a specific Γ, and in this case it will simply be a regular simplicial neighbourhood with respect to some Γ.

Proof. There exists a simplicial homeomorphism that embeds Δ into the boundary of its regular neighbourhood in a sufficiently fine refinement of M by a folklore general position argument. This map can be assumed to be facewise linear on Δ by [AP25]. A subdivision of M that realizes the regular neighbourhood therefore gives the desired subdivision. $\square$

Corollary 6.10. *In the situation of the previous lemma, there exists in a refinement M' of M a hypersurface with boundary $E_\Delta \supset \Delta$ so that*

$$\mathscr{A}^k(\Delta) \cong \mathscr{A}^k(E_\Delta).$$

We can assume that ∂E_Δ is an induced subcomplex of E_Δ.

We shall improve on this later, but it is a good illustration why reducing to hypersurfaces is enough. We need the following proposition, essentially due to Whiteley [Whi90], see also [BN10, Mur10]:

Proposition 6.11. *Consider a triangulated PL manifold S of dimension $2k-2$, realized in $\Bbbk^{2k-1}$. Consider an element $\ell \in \mathscr{A}^1(S)$, and an edge e of S. Assume that ℓ acts as a middle Lefschetz element on $\mathrm{lk}_e\, S$. Consider the stellar subdivision on the edge, and a generic linear combination ℓ' of ℓ with the variable of the new vertex v_e. Then*

$$\mathrm{coker}[\mathscr{A}^{k-1}(S) \xrightarrow{\cdot \ell} \mathscr{A}^k(S)] \hookrightarrow \mathrm{coker}[\mathscr{A}^{k-1}(S{\uparrow}e) \xrightarrow{\cdot \ell'} \mathscr{A}^k(S{\uparrow}e)].$$

Proof of Corollary 6.10. We can assume that the link of every edge not in E is combinatorially equivalent to a polytope by applying sufficiently many stellar subdivisions [AI15].

If E_Δ supports an element of degree k outside Δ then it must be supported in a $(k-1)$-face outside Δ. Removing that face may affect the manifold property, but we can refine E_Δ and M outside Δ so that the star of that face is a compact ball in the interior of E_Δ, in which case its removal preserves the manifold property.

To make sure this process terminates in finite time, we have to ensure that we do not introduce any new elements of degree k in the refinement; this is ensured by Proposition 6.11 and the Lefschetz theorem for spheres arising as boundaries of polytopes [Sta80]. □

We shall need a stronger version of this corollary later, but we leave it here anyway because it serves to illustrate our strategy.

Let us make two quick remarks:

Remark 6.12. Instead of subdividing first to make links polytopal, we could of course simply use an induction and assume the Lefschetz theorem for $(2k-4)$-spheres already.

Remark 6.13. It is not hard to see that subdivisions at other faces similarly preserve the Lefschetz property (see [BN10]), but for simplicity we restrict to subdivisions of edges.

We call such a complex, that is, a supercomplex E of Δ such that $\mathscr{A}^k(\Delta) \cong \mathscr{A}^k(E)$ holds, an **envelope** in degree k for Δ. Passing from a subcomplex of M to its envelope (in degree k) does not affect the biased Poincaré duality (in degree k) as in such case

$$\mathscr{K}^k(M,\Delta) \ \cong\ \mathscr{K}^k(M,E_\Delta).$$

Using the above, we can now replace any $(k-1)$-complex, such as the skeleton of a triangulated sphere, by a hypersurface. How does this help? First, we shall now see that a Lefschetz property for that hypersurface is equivalent to the desired biased Poincaré duality property. All known generic Lefschetz theorems and their proofs rely on this fact, see also Section 10.2.

6.7. Characterization theorem for closed hypersurfaces.

We next want to deal with envelopes that are closed hypersurfaces. After all, we just argued that we can replace $(k-1)$-dimensional complexes in $(2k-1)$-dimensional manifolds with hypersurfaces (though, be careful, not generally closed). We nevertheless offer up this first intermediate case for rather didactic purposes; it is less technical than the case of hypersurfaces with boundary examined next.

We are almost in the same situation as in Proposition 4.6: M is a manifold of dimension $2k-1 = d-1$ in $\mathbb{k}^d$, but this time E is a closed hypersurface in it that is not necessarily a sphere. For simplicity, we shall assume such a hypersurface to be relatively $(k-2)$-acyclic (of course, with respect to $\mathbb{k}$), that is, the map $E \hookrightarrow M$ is injective in $\mathbb{k}$-homology up to dimension $k-2$. We will see that this is sufficiently general for our purposes. Indeed, following the proof of Proposition 6.9, this is a valid assumption we can make for PL manifolds: every Δ of dimension $k-1$ in M lies in a relatively $(k-2)$-acyclic closed hypersurface after subdivisions of M outside of Δ. We shall also assume that the map $E \hookrightarrow M$ is the trivial map in homology in dimension $d-2$, and shall treat the remaining case using the culminating methods of the next Section 6.8.

Note that then M is parted into a finite number of compact simplicial manifolds $(M_i)_{i\in I}$ such that the union of their boundaries is the hypersurface E. Note that if $k>1$, then there are at most two of these manifolds. Note further that biased Poincaré duality with respect to E is equivalent to biased Poincaré duality with respect to

$$\overline{M}_j \ = \ \bigcup_{i\in I\setminus\{j\}} M_i$$

for all $j\in I$ if $|I|\geq 2$ as the ideals $\mathscr{K}(M,\overline{M}_j)$ are orthogonal to each other in the Poincaré pairing.

We may therefore restrict to examining $\overline{M}_1$ without loss of generality, and may assume that

$$E \ = \ \partial M_1 \ = \ \partial \overline{M}_1.$$

Theorem 6.14. *If $|I|\geq 2$, then a closed orientable $(d-1)$-manifold M in $\Bbbk^d$ satisfies biased Poincaré duality with respect to $\overline{M}_1$ and degree $k = {}^d\!/_2$ as above if and only if the sequence*

$$(H^{k-1})^{\binom{d}{k}}(M) \ \longrightarrow \ (H^{k-1})^{\binom{d}{k}}(M_1) \ \longrightarrow \ \mathscr{B}^k(E\subset M) \ \longrightarrow \ 0$$

is exact.

The proof is a direct combination of Proposition 4.6 and Corollary 6.2.

Proof. The theorem follows at once from the commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \widetilde{\mathscr{K}}^k(M_1,E) \arrow[d] \arrow[dr, dashed] & & \\
(H^{k-1})^{\binom{d}{k}}(M) \arrow[r] & (H^{k-1})^{\binom{d}{k}}(M_1) \arrow[r] \arrow[dr, dashed, "\mathrm{hom}"] & \mathscr{B}^k(M_1\subset M) \arrow[r] \arrow[d] & \mathscr{B}^k(M_1) \arrow[r] & 0 \\
 & & \mathscr{B}^k(E\subset M) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

where the top space $\widetilde{\mathscr{K}}^k(M_1,E)$ is defined by the vertical exact sequence. It injects into $\mathscr{K}^k(M,\overline{M}_1)$, which is the Poincaré dual to $\mathscr{B}^k(M_1)$ and therefore is isomorphic to it as vector space. Hence, the exact sequence of the assumption is equivalent to an isomorphism

$$\mathscr{B}^k(M_1) \cong \ \widetilde{\mathscr{K}}^k(M_1,E) \ \twoheadleftarrow \ \mathscr{K}^k(M,\overline{M}_1),$$

where the spaces on the extreme left and extreme right are of the same dimension. Hence, the first isomorphism is therefore equivalent to an isomorphism between the extreme spaces, which is exactly biased Poincaré duality by Corollary 6.2. $\square$

6.8. **The doubling trick and the general characterization theorem.** To prove biased Poincaré duality for general subcomplexes, the above criterion will not be enough, unfortunately. In particular, we cannot always assume that E is a closed hypersurface. Instead, the criterion below will suffice in general by Corollary 6.10.

Again, we are almost in the same situation as before, gradually generalizing until we can now formulate the main theorem: M is again a (closed, orientable) manifold of dimension $2k-1=d-1$ in $\mathbb{k}^d$, but this time around we allow for hypersurfaces in it to have boundary.

Consider for this purpose the complement $\widetilde{M}$ of E in M, where E is a relatively $(k-2)$-acyclic orientable hypersurface in M whose boundary ∂E is an induced subcomplex of E, and the double $\mathrm{D}E$ of E. Notice that the proof of Corollary 6.10 guarantees the existence of such an envelope in degree k for any $(k-1)$-dimensional subcomplex of a combinatorial M (potentially having to subdivide M outside of that complex).

There is a **folding map**

$$\tau : \mathrm{D}E \ \longrightarrow\ E$$

for doubles that will be tremendously useful, which identifies the two copies of E that we call the **charts** of $\mathrm{D}E$.

The open manifold $\widetilde{M}$ can be compactified canonically to a compact manifold $\widetilde{\mathrm{D}}M$ with boundary $\mathrm{D}E$ (since E was orientable), and the folding map extends to a map

$$\tau : \widetilde{\mathrm{D}}M \ \longrightarrow\ M.$$

Let us also introduce the following useful concept: Let π denote the general position projection to a hyperplane H, and ϑ the height over that projection, so that

$$\mathscr{A}^k(X) \ =\ {\mathscr{A}^k(\pi X)}\Big/{\vartheta \mathscr{A}^{k-1}(\pi X)}$$

for a complex X in $\mathbb{k}^d$. Let us first observe the following helpful lemma that translates the problem to the case of doubles. It, in itself, is using Lefschetz properties.

Lemma 6.15. *Consider π a general position projection to a hyperplane, and let ϑ denote the height over that hyperplane. The following are equivalent:*

(1) The map

$$\mathscr{B}^{k-1}(\pi E, \partial \pi E) \ \xrightarrow{\cdot\vartheta}\ \mathscr{B}^k(\pi E \subset \pi M),$$

which factors as

$$\begin{tikzcd}
\mathscr{B}^{k-1}(\pi E, \partial \pi E) \arrow[r] \arrow[d] & \mathscr{B}^k(\pi E \subset \pi M) \\
\mathscr{B}^{k-1}(\pi E \subset \pi M) \arrow[ur, "\cdot\vartheta"] &
\end{tikzcd}$$

is an injection.

(2) The isomorphism

$$\mathscr{A}^k(M, E) \ \xrightarrow{\sim}\ \mathscr{A}^k(\widetilde{\mathrm{D}}M, \mathrm{D}E)$$

descends to an isomorphism of the image $\mathscr{K}^k(M,E)$ *of*

$$\mathscr{A}^k(M,E) \longrightarrow \mathscr{B}^k(M).$$

to the image $\mathscr{K}^k(\widetilde{\mathrm{D}}M,\mathrm{D}E)$ *of*

$$\mathscr{A}^k(\widetilde{\mathrm{D}}M,\mathrm{D}E) \longrightarrow \mathscr{B}^k(\widetilde{\mathrm{D}}M \subset M).$$

Here we read $\mathscr{B}^k(X \subset M)$ for $X \subset \widetilde{\mathrm{D}}M$ as the cokernel of the composition

$$\begin{tikzcd}
(H^{k-1})^{\binom{d}{k}}(M) \arrow[r] \arrow[d, "\tau"] & \mathscr{A}^k(X) \\
(H^{k-1})^{\binom{d}{k}}(\widetilde{\mathrm{D}}M) \arrow[ur] &
\end{tikzcd}$$

Moreover, let us note that for a manifold X with boundary ∂X, we define $\mathscr{B}(X,\partial X)$ naturally using the definition of stars in relative complexes:

$$\mathrm{st}_v(X,\partial X) \;=\; (\mathrm{st}_v X, \mathrm{st}_v \partial X).$$

Let us briefly take a moment to explain why we require the above lemma, for all its technicality: We wish to prove the existence of an injection

$$\mathscr{B}^k(M)_{|\overline{E}} \hookrightarrow \mathscr{K}^k(M,E).$$

Unfortunately, the first space, a partitioned quotient of $\mathscr{B}^k(M)$, lacks a nice description. Therefore, we look at doubles. In a déjà-vu from Proposition 6.6, notice that

$$\mathscr{A}^k(\mathrm{st}_\sigma \widetilde{\mathrm{D}}M) \;=\; 0$$

unless σ is an interior $(k-1)$-face of $\widetilde{\mathrm{D}}M$, so that we have a well-defined map

$$\bigoplus_{\substack{\sigma\in M^{(k-1)},\\ \sigma\notin E}} \mathscr{A}^k(\mathrm{st}_\sigma M) \longrightarrow \bigoplus_{\sigma\in \widetilde{\mathrm{D}}M^{(k-1)}} \mathscr{A}^k(\mathrm{st}_\sigma \widetilde{\mathrm{D}}M)$$

and in particular a map

$$\mathscr{B}^k(\widetilde{\mathrm{D}}M) \longrightarrow \mathscr{B}^k(M)_{|\overline{E}}.$$

We can now see how the doubling construction helps: Consider the commutative diagram

$$\begin{tikzcd}
\mathscr{K}^k(M,E) \arrow[r] \arrow[d] & \mathscr{K}^k(\widetilde{\mathrm{D}}M,\mathrm{D}E) \arrow[d] \\
\mathscr{B}^k(M)_{|\overline{E}} \arrow[r] \arrow[d, hook] & \mathscr{B}^k(\widetilde{\mathrm{D}}M) \arrow[d, hook] \\
\bigoplus_{\substack{\sigma\in M^{(k-1)},\\ \sigma\notin E}} \mathscr{A}^k(\mathrm{st}_\sigma M) \arrow[r] & \bigoplus_{\sigma\in \widetilde{\mathrm{D}}M^{(k-1)}} \mathscr{A}^k(\mathrm{st}_\sigma \widetilde{\mathrm{D}}M).
\end{tikzcd}$$

The previous lemma ensures that the top horizontal map is an injection. Hence, proving an injection of the left top vertical map is reduced to proving an injection on the right top vertical map.

Proof of Lemma 6.15. We prove that (1) implies (2); this is, for our purposes, the critical direction. The converse is obtained by reversing the logic of the steps below.

The map described in (2) is clearly a surjection as it is the specialization of a surjective map. But the condition of (1) ensures that the kernels A of

$$\mathscr{B}^{k-1}(\pi E \subset \pi M) \xrightarrow{\cdot\vartheta} \mathscr{B}^{k}(\pi E \subset \pi M)$$

and B of

$$\mathscr{B}^{k-1}(\pi \mathrm{D} E \subset \pi M) \xrightarrow{\cdot\vartheta} \mathscr{B}^{k}(\pi \mathrm{D} E \subset \pi M)$$

inject into $\mathscr{A}^{k-1}(\partial\pi E)$, which in particular implies a surjection (and in fact an isomorphism)

$$A \longrightarrow\!\!\!\!\!\rightarrow B.$$

On the other hand, we have by relative acyclicity the following commutative diagram for degrees at most k:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{A}^*(\pi\widetilde{\mathrm{D}}M, \pi\mathrm{D}E) \arrow[r] & \mathscr{B}^*(\pi\widetilde{\mathrm{D}}M \subset \pi M) \arrow[r] & \mathscr{B}^*(\pi\mathrm{D}E \subset \pi M) \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{A}^*(\pi M, \pi E) \arrow[r] \arrow[u] & \mathscr{B}^*(\pi M) \arrow[r] \arrow[u] & \mathscr{B}^*(\pi E \subset \pi M) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

which after reduction by ϑ implies a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] & \mathscr{A}^k(\widetilde{\mathrm{D}}M, \mathrm{D}E) \arrow[r] & \mathscr{B}^k(\widetilde{\mathrm{D}}M \subset M) \\
0 \arrow[r] & A \arrow[r] \arrow[u, two heads] & \mathscr{A}^k(M, E) \arrow[r] \arrow[u] & \mathscr{B}^k(M) \arrow[u]
\end{tikzcd}$$

which in turn implies that the desired map

$$\mathscr{A}^k(M, E) \longrightarrow \mathscr{A}^k(\widetilde{\mathrm{D}}M, \mathrm{D}E)$$

is also a surjection when restricted to images in $\mathscr{B}^k(M)$ and $\mathscr{B}^k(\widetilde{\mathrm{D}}M \subset M)$, respectively. □

This takes us rather far, as we now have to prove that $\mathscr{B}(\widetilde{\mathrm{D}}M)$ is the image of $\mathscr{K}(\widetilde{\mathrm{D}}M, \mathrm{D}E)$ in degree k. We combine this with the arguments for Theorem 6.14 to obtain:

Theorem 6.16. *In the situation of the previous Lemma 6.15,*

$$(H^{k-1})^{\binom{d}{k}}(M) \longrightarrow (H^{k-1})^{\binom{d}{k}}(\widetilde{\mathrm{D}}M) \longrightarrow \mathscr{B}^k(\mathrm{D}E \subset M)$$

is exact if and only if

$$\mathscr{K}^k(\widetilde{\mathrm{D}}M, \mathrm{D}E) \longrightarrow \mathscr{B}^k(\widetilde{\mathrm{D}}M)$$

is injective.

Proof. Consider as in the proof of Theorem 6.14 the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathscr{K}^k(\widetilde{\mathrm{D}}M, \mathrm{D}E) \arrow[d] \arrow[dr, dashed] & & \\
(H^{k-1})^{\binom{d}{k}}(M) \arrow[r] & (H^{k-1})^{\binom{d}{k}}(\widetilde{\mathrm{D}}M) \arrow[r] \arrow[dr, dashed, "\mathrm{hom}"] & \mathscr{B}^k(\widetilde{\mathrm{D}}M \subset M) \arrow[r] \arrow[d] & \mathscr{B}^k(\widetilde{\mathrm{D}}M) \arrow[r] & 0 \\
 & & \mathscr{B}^k(\mathrm{D}E \subset M) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

The vertical sequence is exact by definition, and the horizontal sequence is exact by Proposition 4.6. The claim follows at once by considering the diagonal maps. $\square$

This is quite useful. Notice the following immediate corollary and extension of Lemma 6.4. We define E as in the setup of this section.

Proposition 6.17. *A $\Bbbk$-PL homeomorphism $\varphi : M \to M'$ of manifolds $M,\ M'$ in $\Bbbk^d$ that restricts to the identity on a common subcomplex E preserves biased Poincaré duality at E, that is, $\mathscr{K}^*(M, \Delta)$ satisfies biased Poincaré duality (in degree k) if and only if $\mathscr{K}^*(M', \Delta)$ does (in degree k).*

6.9. **The HGL relations.** Finally, biased Poincaré duality allows us to formulate a Lefschetz property at ideals. We say that M a manifold of dimension $d-1$ in $\Bbbk^d$ satisfies the **HGL relations** in degree $k \le \frac{d}{2}$ and with respect to an ideal $\mathscr{I}^* \subset \mathscr{B}^*(M)$ if there exists an ℓ in $\mathscr{B}^1(M)$, the pairing

$$\begin{array}{ccccc}
\mathscr{I}^k & \times & \mathscr{I}^k & \longrightarrow & \mathscr{I}^d \\
a & & b & \longmapsto & \mathrm{vol}(ab\ell^{d-2k})
\end{array} \tag{9}$$

is nondegenerate. Usually interested in face-rings, we will say M satisfies the **HGL relations** if it does so at all squarefree monomial ideals, that is, for all $\mathscr{K}^*(M, \Delta)$, where Δ is any subcomplex of M, and their annihilators. As $\mathscr{K}^*(M, \Delta)$ is generated by monomials not in Δ, these are described as

$$\overline{\mathscr{K}}^*(M, \Delta) \ \coloneqq\ \ker \left[\mathscr{B}^*(M) \ \longrightarrow \bigoplus_{\substack{\sigma \in M \\ \sigma \notin \Delta}} \mathscr{A}^*(\mathrm{st}_\sigma M) \right].$$

Notice that the two statements are equivalent for the middle pairing, that is, when $2k = d$, because $\mathscr{K}^k(M, \Delta)$ and $\overline{\mathscr{K}}^k(M, \Delta)$ are orthogonal complements in $\mathscr{B}^k(M)$. Biased Poincaré duality is evidently a weaker form of the HGL relations, with equivalence in the case $k = d - k$.

7. Perturbations and Lefschetz elements via biased Poincaré duality

To prove biased Poincaré duality in higher degrees, we need to understand the Lefschetz theorem for face rings of manifolds; this in itself is not a hard task, but requires some care as to how we actually construct Lefschetz elements. The idea is provided in the following lemma, that we shall explore throughout the section.

7.1. **The Kronecker perturbation lemma.**

The following lemma is, together with biased Poincaré duality, the heart of this paper:

Lemma 7.1. *Consider two linear maps*

$$\alpha, \beta : \mathscr{X} \longrightarrow \mathscr{Y}$$

of two vector spaces $\mathscr{X}$ and $\mathscr{Y}$ over $\Bbbk$.

(1) Assume that β has image transversal to the image of α, that is,

$$\beta(\ker\alpha) \cap \operatorname{im}\alpha = 0 \subset \mathscr{Y}.$$

Then a generic linear combination α"+"β of α and β has kernel

$$\ker(\alpha \text{ ``+'' } \beta) = \ker\alpha \cap \ker\beta.$$

(2) Similarly, if

$$\beta^\diamondsuit(\ker(\alpha^\diamondsuit)) \cap \operatorname{im}(\alpha^\diamondsuit) = 0 \subset \mathscr{X}^\diamondsuit$$

or equivalently

$$\beta^{-1}(\operatorname{im}\alpha) + \ker\alpha = \mathscr{X},$$

then

$$\operatorname{im}(\alpha \text{ ``+'' } \beta) = \operatorname{im}\alpha + \operatorname{im}\beta.$$

Here we use $(\cdot)^\diamondsuit$ to denote dual maps and vector spaces in the basic linear-algebra sense of the notion.

The lemma is probably classical, but was at least known to Kronecker [Kro90], see also [Rin13] for a statement in form of representation theory of the Kronecker quiver.

This is a surprisingly simple yet powerful lemma, as it allows us to control the kernel resp. the image of a linear combination of two maps. Note that the conditions are naturally dual to each other, so that when we apply the construction in Gorenstein rings associated to simplicial manifolds, we usually get both conclusions for the price of one. Let us also remark that the proofs of the Lefschetz property via the decomposition principle, such as those of Murai and Whiteley (Proposition 6.11) discussed earlier, can be thought of as employing a very special case of the perturbation lemma.

7.2. **The transversal prime property.**

The strategy for the Lefschetz property, proven via the biased pairing property, is to prove the following property:

Let M denote a $(d-1)$-manifold (possibly with boundary) in $\Bbbk^d$, and $k \leq \frac{d}{2}$. Let W denote a subset of the vertices of M. Then M has the **transversal prime property** in degree k and with respect to

W if

$$\begin{aligned}&\ker\left[\left(\text{“}\sum_{w\in W}\text{”}x_w\right)^{d-2k}: \mathscr{B}^k(M)\ \longrightarrow\ \mathscr{B}^{d-k}(M,\partial M)\right]\\ &= \mathrm{ann}_{\mathscr{B}^k(M)}\left\langle x_w | w\in W\subset M^{(0)}\right\rangle\\ &= \bigcap_{w\in W}\ker\left[x_w: \mathscr{B}^k(M)\ \longrightarrow\ \mathscr{B}_{k+1}(M)\right]\end{aligned}$$

We remind the reader that

$$\text{“}\sum_{w\in W}\text{”}x_w$$

stands for a generic linear combination of indeterminates x_w with $w\in W$.

Note that if the transversal prime property holds for $W = M^{(0)}$, then we conclude the hard Lefschetz theorem for M. The general strategy for us is to prove the transversal prime property by linearly ordering the vertices in any way, and proceeding by induction on the size of initial segments.

7.3. **The perturbation approach.** Consider M a manifold with boundary of dimension $(d-1)$ realized in $\Bbbk^d$, and assume that we proved the transversal prime property for a set of vertices W' of M for the map between degree $d-k$ and degree k. For simplicity of notation, we will assume that $d = 2k+1$, the other Lefschetz maps will be reduced to that case. Consider an additional vertex of M not in W'.

When we want to prove the transversal prime property for $W = W'\cup\{w\}$, we want to use Lemma 7.1, applied to the maps

$$\alpha\ =\ \text{“}\sum_{v\in W'}\text{”}x_v \quad \text{and} \quad \beta\ =\ x_w$$

and to the spaces

$$\mathscr{X}\ =\ \mathscr{B}^k(M) \quad \text{and}\ \mathscr{Y}\ =\ \mathscr{B}^{k+1}(M,\partial M).$$

Now, note that if we have already proven the transversal prime property with respect to W', then kernel and image of α have a particularly nice description:

$$\ker\alpha\ =\ \bigcap_{v\in W'}\ker\left[x_v: \mathscr{B}^k(M,\partial M)\ \longrightarrow\ \mathscr{B}^{k+1}(M)\right]$$

and

$$\mathrm{im}\alpha\ =\ \mathrm{im}\left[\bigoplus_{v\in W'}\mathscr{A}^k(\mathrm{st}_v M)\longleftarrow \mathscr{B}^k(M)\right].$$

We have the following central theorem:

Theorem 7.2. *In the situation above, the following are equivalent.*

(1) The assumptions of Lemma 7.1(1) hold, that is,

$$\beta(\ker\alpha)\ \cap\ \mathrm{im}\alpha\ =\ 0\ \subset\ \mathscr{Y}.$$

(2) The assumptions of Lemma 7.1(2) hold, that is,

$$\beta^{-1}(\mathrm{im}\alpha)\ +\ \ker\alpha\ =\ \mathscr{X}.$$

(3) The pullback of $\ker\alpha$ *to*

$$\mathscr{A}^k(\mathrm{st}_w M)\ \cong\ \mathscr{A}^k(\mathrm{lk}_w\, M)$$

along x_w *satisfies biased Poincaré duality in the* $(d-2)$*-sphere* $\mathrm{lk}_w\, M$*.*

Proof. We have $(1) \Leftrightarrow (2)$ by Poincaré duality. For $(3) \Leftrightarrow (1)$, observe that $\ker\alpha$ and $\mathrm{im}\alpha$ are orthogonal complements in $\mathscr{B}^*(M)$. Hence

$$\mathscr{I}_W\ :=\ \left(\mathrm{im}\left[\mathscr{B}^k(M,\partial M)\ \xrightarrow{\text{“}\sum_{v\in W'}\text{”}x_v}\ \mathscr{B}^{k+1}(M)\right]\right)\ \cap\ \mathscr{A}^k(\mathrm{st}^\circ_w M)$$

in $\mathscr{A}^k(\mathrm{st}_w M)$ and

$$\mathscr{K}_W\ :=\ x_w\alpha\ \subset\ \mathscr{A}^k(\mathrm{st}_w M)$$

are orthogonal to each other. It follows that

$$x_w\ker\alpha\ \cap\ \mathrm{im}\alpha\ =\ 0\ \ \text{in}\ \mathscr{B}^k(M,\partial M)$$

if and only if the biased pairing property holds for $\mathscr{K}_W$ in $\mathscr{A}^*(\mathrm{st}_w M)$ in degree k.

To finally show that also $(1) \Rightarrow (3)$, it suffices to show

Lemma 7.3. *$\mathscr{I}_W$ and $\mathscr{K}_W$ are orthogonal complements in $\mathscr{A}^k(\mathrm{lk}_w\, M)$.*

For this, it suffices to argue that their dimensions sum up to $\dim \mathscr{A}^k(\mathrm{st}_w M)$, which follows by the short exact sequence

$$0\ \longrightarrow\ \mathscr{A}^k(\mathrm{st}_w M)\ \longrightarrow\ \mathscr{B}^{k+1}(M)\ \longrightarrow\ \mathscr{B}^{k+1}(M-w\subset M)\ \longrightarrow\ 0$$

and because

$$\ker\alpha\cap\ker x_w\quad\text{and}\quad \mathrm{im}\alpha\ +\ \mathrm{im}x_w$$

are orthogonal complements in $\mathscr{B}^{k+1}(M-w\subset M)$ and its Poincaré dual in $\mathscr{B}^k(M,\partial M)$ □

We can therefore extend the transversal prime property from W' to W if any of the above conditions hold. Note further that the task to verify the conditions in Theorem 7.2 is much simpler if we use the transversal prime property applied to W', which we did not assume for its proof (indeed, we could have chosen α in any other way). In that case, the kernel of α has a nice description by induction, and so does its image.

This has an especially nice description if $M-W'$ and

$$\bigcup_{v\in W'} \mathrm{st}_v M$$

are codimension zero manifolds, when the full force of Proposition 4.6 applies. As we see, the pullbacks to the star of w appear directly as $\mathscr{K}_W$ resp. $\mathscr{I}_W$ above in the induction step, this is of central importance for our calculations. Unfortunately, this is not always the case. But we can force it, using a construction from the next section.

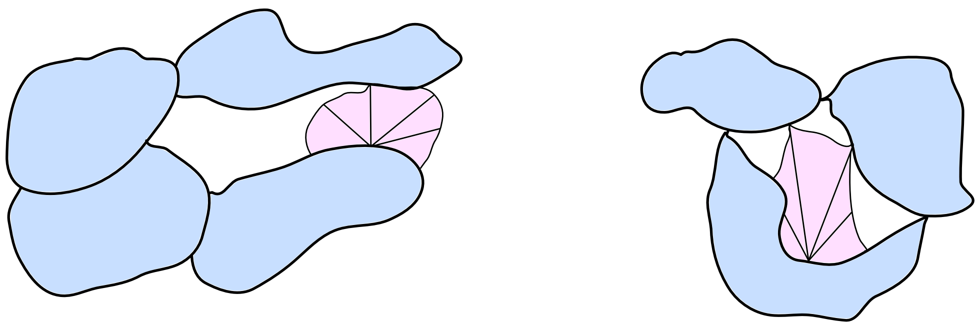

Figure 7.3. The perturbation approach is more approachable if the intermediate complexes are manifolds

7.4. **Weakly deconstructible spheres and balls.** We illustrate the perturbation approach by proving the Lefschetz property for a larger class of spheres (and balls), which nevertheless can be decomposed nicely enough. Again, as natural, "ball" here means $\mathbb{k}$-homology manifold with the $\mathbb{k}$-homology of a ball.

To define this, we say that for a 0-dimensional complex, the removal of any vertex is a valid **weak deconstruction step**. Hence, any 0-dimensional complex is weakly deconstructible.

Consider now a $(d-1)$-dimensional ball or sphere X, and a vertex v in it with link $Y = \mathrm{lk}_v X$. Consider X_v the subcomplex induced by all facets not containing v.

We say that the transition from X to X_v is a **weak deconstruction step** if $\mathrm{lk}_v X_v$ is a weakly deconstructible $(d-2)$-ball or sphere or empty and obtained from $\mathrm{lk}_v X$ by a sequence of weak deconstruction steps, and finally $\partial \mathrm{lk}_v X_v$ is a weakly deconstructible $(d-3)$-sphere if nonempty. A ball or sphere is **weakly deconstructible** if it can be reduced to the empty complex by weak deconstruction steps.

Theorem 7.4. *Consider a ball or sphere X of dimension $d-1$, and a sequence of weak deconstruction steps $V = (v_1, \cdots, v_n)$. Consider further a generic Artinian reduction $\mathscr{A}^*(X)$ of the face ring of X, and a generic ℓ supported on the vertices (v_i). Let finally X_V denote the ball induced by the remaining facets.*

Then, for all $k \le d/2$, the map

$$\mathscr{A}^k(X, \partial X) \xrightarrow{\ \cdot \ell^{d-2k}\ } \mathscr{A}^{d-k}(X).$$

has kernel $\mathscr{A}^k(X_V, \partial X_V)$.

Proof. Following the general perturbation approach, it suffices to consider $V = (v)$, that is, a single weak deconstruction step. Consider the link Y of $v \in X$, a projection π to $\mathbb{k}^{d-1}$ and ϑ the height over that projection. We prove the theorem by induction on the dimension.

We have to establish that the pullback $\mathscr{J}$ of $\mathscr{A}^k(X, \partial X)$ to $\mathscr{A}^k(\pi Y, \pi \partial Y)$ via the cone lemma satisfies the appropriate HGL relation: the pairing

$$\begin{array}{rccccc} \mathrm{Q}_{\mathscr{J},k,\vartheta}: & \mathscr{J}^k & \times & \mathscr{J}^k & \longrightarrow & \mathscr{J}^{d-1} \\ & a & & b & \longmapsto & \mathrm{vol}(ab\vartheta^{d-2k-1}) \end{array} \tag{10}$$

is nondegenerate in $\mathscr{A}^k(\pi Y, \pi\partial Y)$. Recall that, by assumption, $Y_v = \mathrm{lk}_v\, X_v$ is obtained from $Y = \mathrm{lk}_v\, X$ by a sequence of weak deconstruction steps, which weakly deconstructs as well. Let $W = (w_1, \cdots, w_m)$ and $W' = (w'_1, \cdots, w'_n)$ be the corresponding sequence of vertices. We can assume that ϑ in Y is supported in $W \cup W'$ only; after all, we only make the linear system more special, and making it more generic later does not lose us the desired Lefschetz properties.

We decompose further and set $\vartheta = \vartheta_W + \vartheta_{W'}$. Let us initially also define $\vartheta_\lambda = \vartheta_W + \lambda\vartheta_{W'}$ for a scalar λ. Let us consider ϑ_0 first.

We can decompose $\mathscr{J} = \mathscr{J}_W \oplus \mathscr{J}'_W$, where $\mathscr{J}'_W = \mathscr{A}^k(\pi Y_v, \pi\partial Y_v)$ and $\mathscr{J}_W = \vartheta_0 \mathscr{A}^{k-1}(\pi Y, \pi Y_v \cup \partial Y)$. This decomposition is orthogonal under $\mathrm{Q}_{\mathscr{J},k,\vartheta_0}$, and by induction assumption, $\mathrm{Q}_{\mathscr{J},k,\vartheta_0}$ is nondegenerate because

$$\mathscr{A}^{k-1}(\pi Y, \partial Y) \xrightarrow{\cdot\vartheta_0^{d-2k+1}} \mathscr{A}^{d-1-(k-1)}(\pi Y).$$

has kernel $\mathscr{J}'_W$.

Moreover, we have that $\vartheta_{W'}$ induces an isomorphism

$$\mathscr{A}^k(\pi Y_v, \pi\partial Y_v) \xrightarrow{\cdot\vartheta_0^{d-2k-1}} \mathscr{A}^{d-1-(k)}(\pi Y_v).$$

Hence, the Kronecker Lemma 7.1 gives the desired nondegeneracy of the HGL pairing for $\vartheta_\lambda = \vartheta_W + \lambda\vartheta_{W'}$, and therefore for ϑ itself. □

Notice that in the induction step, we actually proved something slightly stronger. We proved:

Lemma 7.5. *Consider a ball or sphere X of dimension $d-1$, and a sequence of weak deconstruction steps $V = (v_1, \cdots, v_m)$, followed by a sequence of weak deconstruction steps $V' = (v'_1, \cdots, v'_m)$ that form a weak deconstruction of X. Consider further a generic Artinian reduction $\mathscr{A}^*(X)$ of the face ring of X, and a generic ℓ supported on the vertices $V \cup V'$. Let finally X_V denote the ball induced by the facets not containing V. Then the pairing*

$$\begin{array}{ccccc} \mathscr{A}^k(X, \partial X) & \times & \mathscr{A}^k(X, \partial X) & \longrightarrow & \mathscr{J}^{d-1} \\ a & & b & \longmapsto & \mathrm{vol}(ab\ell^{d-2k-1}) \end{array} \tag{11}$$

is nondegenerate on

$$\ell\mathscr{A}^{k-1}(X, \partial X) \cup \mathscr{A}^k(X_V, \partial X_V).$$

While still rather restrictive, the next sections show that proving the Lefschetz property for weakly deconstructible spheres and balls is all that is needed to prove the Lefschetz properties for all manifolds.

8. Railway construction

The plan now should be clear. For simplicity, let us first assume that M is a sphere of even dimension $2k$.

If there is a total order on the vertices of M so that $\mathrm{N}_i M$, the union $\bigcup_{v\in\{1,\cdots,i\}} \mathrm{st}_v\, M_i$, is "nice enough" for all i (which we will define later), then we apply the approach of Section 7.3 directly.

The issue is of course what happens if this is not the case: In this case, we embed M into its suspension $\operatorname{susp} M$. Since we want to prove the isomorphism

$$\mathscr{A}^k(M) \xrightarrow{\cdot \ell} \mathscr{A}^{k+1}(M)$$

we only care about the k-skeleton of M, which we denote by S for the moment. Assume now, after subdivisions that do not affect $S \subset \operatorname{susp} M$, the subcomplex S has an envelope E which *does* have a "nice enough" decomposition. Then we can apply Section 7.3. Of course, to say something about the biased pairing property at S, we want to prove a statement about the double of E, and apply Theorem 6.16. But that, it turns out in the next section, is just a Lefschetz property for E itself.

Before we introduce it, let us pin down what we mean by a "nice enough" decomposition, which locally incorporates the notion of weak deconstructible balls and spheres.

8.1. **Métros.** We differ here from [Adi18] with a different, and simpler construction.

We say an orientable rational manifold U in $\mathbb{k}^d$ with induced boundary and of dimension $d-1$, and Δ a $(k-1)$-dimensional induced subcomplex of the interior $U^\circ = (U, \partial U)$ of U, and finally a linear order on the vertices of Δ is a **railway** in degree $k \leq d$ if the following conditions are satisfied:

(1) For every initial segment W of the vertices of Δ, the complex $\mathrm{N}_W U$ is a manifold that collapses onto $\mathrm{N}_W \Delta$.
(2) U is an envelope for Δ in degree k: we have $\mathscr{A}^k(U) = \mathscr{A}^k(\Delta)$.

We call Δ its **tabula**. We call U a **métro** if it satisfies Condition (1), but not necessarily (2).

It is often useful to impose an additional condition of topological simplicity. We call a railway resp. métro **octavian** if:

(1) $\mathrm{N}_W U$ in U is $(k-2)$-acyclic for all initial segments W, and
(2) so is its complement in U: the remaining facets to $\mathrm{N}_W U$ in U form a manifold $\overline{U}_W$ in U and it is $(k-2)$-acyclic.

We call a (octavian) métro of degree k **hereditary** if for every initial segment W with final vertex w, and $W' = W \setminus (w)$,

(1) the complex $\operatorname{lk}_w \overline{U}_{W'}$ is a weakly deconstructible complex, of which $(\partial \overline{U}_{W'} \cup \mathrm{N}_W U) \cap \operatorname{lk}_w \overline{U}_{W'}$ is an initial segment, that is, $(\partial \overline{U}_{W'} \cup \mathrm{N}_W U) \cap \operatorname{lk}_w \overline{U}_{W'}$ gets removed first in the weak reconstruction sequence and
(2) $(\mathrm{N}_W U) \cap \operatorname{lk}_w \overline{U}_{W'}$ is an initial segment (not necessarily of the same weak deconstruction sequence).

Lemma 8.1. *Consider the k-skeleton S of a $2k$-sphere M as a subcomplex of $\operatorname{susp} M$. Then there is a $\mathbb{k}$-homeomorphism $\operatorname{susp} M \to \widehat{M}$ not affecting M such that $\widehat{M}$ contains an octavian hereditary métro for tabula $S \subset M \subset \widehat{M}$.*

We need the following auxiliary lemma. Inspired by Whitehead's notion of shellability [Whi38], we say that a $(d-1)$-manifold M **elementarily** W**-shells** to a manifold M' if $M = M' \cup D$ and $M' \cap D$ is a $(d-2)$-disk. A W**-shelling** is a composition of several elementary W-shelling steps.

Lemma 8.2. *Consider any k-dimensional subcomplex Δ of a d-sphere M as a subcomplex of* $\operatorname{susp} M$, $2k < d$, *and assume the vertices of Δ are totally ordered.*

Let Δ_i denote the subcomplex on the first i vertices. Then there is a $\Bbbk$-homeomorphism $\operatorname{susp} M \to \widehat{M}$ of $\operatorname{susp} M$ not affecting M, and D_i d-dimensional submanifolds of $\widehat{M}$, such that for all i,

(1) D_i is a ($\Bbbk$-homology) ball containing Δ_i, but no other vertices of Δ, and
(2) D_{i+1} shells to D_i.

Proof. The proof is easy if $k = 0, d = 1$. We can therefore assume that $d - k > 1$.

Consider one of the suspension vertices, say $\mathbf{n}$. We consider the cones $C_i = \mathbf{n} * \Delta_i$ in $\operatorname{susp} M$. These are contractible, and therefore their neighborhoods are balls of dimension $d+1$. After subdivisions not affecting M we can arrange these neighborhoods C_i to contain Δ_i in the boundary, but no other vertices of Δ, and so that C_{i+1} is the union of C_i with a homology $(d+1)$-ball C_i', which intersect along a d-ball K_i. Because $d-k > 1$, we can assume that the homology spheres ∂K_i share a common point.

Remove a neighborhood B of that point. Then, if Δ has m vertices, we set $D_m = \partial C_m \setminus B$. We find $D_{m-1} = D_m \setminus (\partial C_m' \setminus K_m)$, and so forth. □

We obtain at once:

Proof of Lemma 8.1. Order the vertices of S in an arbitrary way. By sufficiently many refinements in M outside S, we can make sure the neighborhood N_i of S_i in $\widetilde{M}$, the induced subcomplex on the first i vertices, is almost a manifold: The link of every face in S_i is a sphere. The only place where an issue can arise is a vertex of S adjacent to S_i, but not in it.

The link of such a vertex can fail to be a ball. However, in the suspension of M, this is repaired by Lemma 8.2. Inductively apply the same lemma in links of lower dimensional faces for the hereditary property. The remaining properties are straightforward. □

8.2. **Métros to railways.** It remains to ensure the envelope property to transition from métros to railways.

Proposition 8.3. *Consider the k-skeleton S of a $2k$-sphere M as a subcomplex of* $\operatorname{susp} M$. *Then there is a $\Bbbk$-homeomorphism $\operatorname{susp} M \to \widehat{M}$ of $\operatorname{susp} M$ that doesn't affect M and contains a hereditary octavian railway for S.*

For the proof, we only need the following final lemma, generalizing the ideas of Corollary 6.10:

Lemma 8.4. *Consider a k-dimensional subcomplex Δ of a $2k+1$-sphere M. Assume E is a hypersurface of M containing Δ, and that E has a simplicial collapse onto Δ (see* [Whi38]*). Then some subdivision E' of E that doesn't affect Δ is an envelope for Δ.*

Since regular neighborhoods collapse onto the subcomplexes they are neighborhoods of, we therefore obtain Proposition 8.3. It remains to prove the Lemma.

Proof. Consider an elementary collapse $E \searrow_e E'$, along a free face σ. Stellarly subdivide at σ, and move the new vertices into general position.

By the weak Lefschetz theorem for $\mathrm{lk}_{e_\sigma} E{\uparrow}\sigma$ (which holds as this is just projective space, that is, the face ring of a simplex), (where e_σ is the unique interior edge to $E{\uparrow}\sigma$ incident to v_σ), we see that after the subdivision, the open star $\mathrm{st}^\circ_{v_\sigma} E{\uparrow}\sigma$ does not support a degree k element: If ϑ is the height over a general position hyperplane in $v^\perp \in \mathbb{k}^d$, π the projection to that hyperplane, then

$$\mathscr{A}^{k-2}(\pi\, \mathrm{lk}_{v_\sigma} E{\uparrow}\sigma) \xrightarrow{\cdot\vartheta} \mathscr{A}^{k-1}(\pi\, \mathrm{lk}_{v_\sigma} E{\uparrow}\sigma)$$

is a surjection.

In particular, $\mathscr{A}^k(\mathrm{st}^\circ_{v_\sigma} E{\uparrow}\sigma)$ is zero by the cone lemma. Hence, any k-stress of $E{\uparrow}\sigma$ must be supported in E'.

Figure 8.4. Subdividing a simple homotopy collapse along a free edge e, and perturbing it to remove stresses in the neighbourhood of the newly created vertex v_e

Repeating this we see that after stellar subdivisions along the collapses, we observe that any degree k element of the railway can be assumed to be generated by the faces of Δ. □

8.3. **Railways in manifolds.** Let us briefly note the more general theorem:

Proposition 8.5. *Consider the k-skeleton S of a $2k$-manifold M as a subcomplex of* $\mathrm{susp}\, M$*. Then there is a $\mathbb{k}$-homeomorphism* $\mathrm{susp}\, M \to \widehat{M}$ *of* $\mathrm{susp}\, M$ *that doesn't affect M and contains a hereditary octavian railway for S.*

Here, the acyclicity in the octavian condition is replaced by relative acyclicity in

$$\mathrm{susp}\, M \setminus \{\text{suspension points}\} \simeq M.$$

The proof is essentially identical, and can be skipped here.

9. Putting things together

Now, again, we put things together. We first treat the case of spheres.

9.1. **Revisiting the characterization theorem.** We return to Section 6.8, but restrict to the case when M is a sphere of dimension $2k-1$. Then E is a $(k-2)$-acyclic orientable hypersurface with boundary.

Lemma 9.1. *Assume that ϑ is supported on the interior vertices of E, and*

$$\mathscr{B}^{k-1}(E) \ \xrightarrow{\cdot\vartheta}\ \mathscr{B}^k(\pi E, \pi\partial E)$$

is an injection.

Then the map

$$(H^{k-1})^{\binom{d}{k}}(\widetilde{\mathrm{D}}M) \ \longrightarrow\ \mathscr{A}^k(\mathrm{D}E) \tag{12}$$

is an injection.

Proof. $\mathrm{D}E$ is realized in $\Bbbk^{[d]}$. We consider $\pi\mathrm{D}E$, realized in $\Bbbk^{[d-1]}$. Then we have an injection

$$(H^{k-1})^{\binom{[d-1]}{k}}(\widetilde{\mathrm{D}}M) \ \hookrightarrow\ (H^{k-1})^{\binom{[d-1]}{k}}(\mathrm{D}E) \ \hookrightarrow\ \mathscr{A}^k(\pi\mathrm{D}E).$$

The cokernel of the righthand map is $\mathscr{B}^k(\pi\mathrm{D}E)$. Notice that the kernel of the map

$$\tau^* : H_{k-1}(\mathrm{D}E) \ \longrightarrow\ H_{k-1}(E)$$

generates $H_{k-1}(\widetilde{\mathrm{D}}M)$, hence $\hom$ embeds $(H^{k-1})^{\binom{[d-1]}{k}}(\widetilde{\mathrm{D}}M)$ into

$$\mathscr{A}^k(\mathrm{D}E) \ =\ \mathscr{A}^k(\pi\mathrm{D}E)\big/\vartheta\mathscr{A}^{k-1}(\pi\mathrm{D}E)$$

and

$$\mathscr{A}^k(\mathrm{D}E)\big/(H^{k-1})^{\binom{[d-1]}{k}}(\widetilde{\mathrm{D}}M) \ \twoheadrightarrow\ \mathscr{A}^k(\mathrm{D}E)\big/(H^{k-1})^{\binom{[d-1]}{k}}(\mathrm{D}E) \ \cong\ \mathscr{B}^k(\pi\mathrm{D}E)\big/\mathscr{B}^{k-1}(\pi\mathrm{D}E)$$

It therefore remains to construct an embedding

$$(H^{k-1})^{\{d\}\cup\binom{[d-1]}{k-1}}(\widetilde{\mathrm{D}}M) \ \hookrightarrow\ \mathscr{B}^k(\pi\mathrm{D}E)\big/\mathscr{B}^{k-1}(\pi\mathrm{D}E).$$

To this end, let us label the two sheets of E in $\mathrm{D}E$ by E_0 and E_1. Consider the subspace $\mathscr{H}$ of elements in $\mathscr{A}^{k-1}(\pi E) = \mathscr{B}^{k-1}(\pi E)$ whose restriction to $\mathscr{A}^{k-1}(\pi\partial E)$ lies in $\hom(H^{k-1})^{\binom{[d-1]}{k-1}}(\partial E)$.

Now, ϑ maps $\mathscr{H} \subset \mathscr{A}^{k-1}(\pi E)$ injectively to $\mathscr{A}^k(\pi E, \pi\partial E)$ by assumption.

Consider therefore an $\alpha \in \mathscr{H}$. We have two copies, represented as $\alpha_0 \in \mathscr{H}_0 \subset \mathscr{A}^{k-1}(\pi E_0)$ and $\alpha_1 \in \mathscr{H}_1 \subset \mathscr{A}^{k-1}(\pi E_1)$, respectively, mapped to $\mathscr{A}^k(\pi E_0, \pi\partial E_0)$ and $\mathscr{A}^k(\pi E_1, \pi\partial E_1)$ by ϑ_0 and ϑ_1, respectively.

We map α to

$$\vartheta_0\alpha_0 - \vartheta_1\alpha_1 \in \mathscr{A}^k(\pi E_0, \pi\partial E_0) \ \oplus\ \mathscr{A}^k(\pi E_1, \pi\partial E_1),$$

which in turn maps to $\mathscr{A}^k(\pi\mathrm{D}E)$.

The image under this map is clearly in the cokernel of

$$\mathscr{B}^{k-1}(\pi\mathrm{D}E) \ \longrightarrow\ \mathscr{B}^k(\pi\mathrm{D}E)$$

Moreover, the image $\widetilde{\mathscr{H}}$ coincides with the image of $(H^{k-1})^{\{d\}\cup\binom{[d-1]}{k-1}}(\widetilde{\mathrm{D}}M)$ in

$$\mathscr{B}^k(\pi\mathrm{D}E)\big/\mathscr{B}^{k-1}(\pi\mathrm{D}E).$$

Combining this with the earlier embedding

$$(H^{k-1})^{\binom{[d-1]}{k}}(\widetilde{\mathrm{D}}M) \ \hookrightarrow\ \mathscr{A}^k(\pi\mathrm{D}E)$$

this gives the desired. $\square$

9.2. **The two Lefschetz properties.** Now, let us assume the conditions of Lemma 6.15 hold, that is,

$$\mathscr{B}^{k-1}(\pi E, \partial\pi E) \xrightarrow{\cdot\overline{\vartheta}} \mathscr{A}^k(\pi E)$$

is an injection.

Then Map (12) is injective if and only if

$$\mathscr{K}^k(\widetilde{\mathrm{D}}M, \mathrm{D}E) \longrightarrow \mathscr{B}^k(\widetilde{\mathrm{D}}M)$$

is injective, which in turn implies the biased pairing property for $\mathscr{K}^k(M, E)$.

So, we observe that we need to prove the first property for our purposes.

Hence, to summarize, to prove the biased pairing property for $\mathscr{K}^k(M, E)$, one has to establish

(1) an injection

$$\mathscr{B}^{k-1}(\pi E, \partial\pi E) \xrightarrow{\cdot\overline{\vartheta}} \mathscr{A}^k(\pi E)$$

and

(2) an injection

$$\mathscr{B}^{k-1}(E) \xrightarrow{\cdot\vartheta} \mathscr{B}^k(\pi E, \pi\partial E)$$

where ϑ is supported on the interior vertices of E.

Note that $\overline{\vartheta}$ and ϑ do not have to coincide. In particular, $\overline{\vartheta}$ does not have to be supported on interior vertices. Because if we have proven the property (2), then we have proven that Map (12) is injective for that particular ϑ, and hence for every generic ϑ (without having to restrict to interior vertices only.)

Note further that these are properties of E, not its double. The latter has a very non-generic linear system of parameters, and proving anything for that case using our techniques is very cumbersome. The system of parameters on E, instead, is rather generic (apart from the condition that ϑ is supported on interior vertices.)

Moreover, the map of (2) is one between Poincaré dual spaces, so it is an injection if and only if it is an isomorphism of Lefschetz type. The first, while not a Lefschetz map as such, can be strengthened to

(1)′ an injection

$$\mathscr{B}^{k-1}(\pi E, \partial\pi E) \xrightarrow{\cdot\vartheta} \mathscr{B}^k(\pi E)$$

and

in which case it, too, is a map between dual spaces and hence equivalent to a Lefschetz isomorphism.

To make it clear then what remains to prove Theorem 1.1 for spheres is to establish these two isomorphisms for octavian, hereditary railways.

9.3. **Proof for spheres.** Consider now a hereditary, octavian railway U in a sphere Σ of dimension $2k-1 = d-1$ realized in $\mathbb{k}^[d]$, which comes with a total order on interior vertices $(v_1, \cdots v_n)$. Our aim is to prove the following:

Lemma 9.2. *Assume the linear system on Σ is in sufficiently general position in $\Bbbk^{[d-1]}$. Then, for generic $\overline{\vartheta}$ and ϑ, the latter supported on the interior vertices of U, we have*

(1)
$$\mathscr{B}^{k-1}(\pi E, \partial\pi E) \ \xrightarrow{\cdot\overline{\vartheta}}\ \mathscr{B}^{k}(\pi E)$$
(2)
$$\mathscr{B}^{k-1}(E) \ \xrightarrow{\cdot\vartheta}\ \mathscr{B}^{k}(\pi E, \pi\partial E).$$

This in particular implies, with the preceding discussion, Theorem 1.1 for spheres of dimension $2k-2$.

Proof. Following the general perturbation approach, it suffices to consider an initial segment W of the vertices of the tabula, and let w be the final vertex of W, and $W' = W \setminus (w)$. We assume by induction that
$$\mathscr{B}^{k-1}(\pi \mathrm{N}_{W'} E, \partial\pi \mathrm{N}_{W'} E) \ \xrightarrow{\cdot\overline{\vartheta_{W'}}}\ \mathscr{B}^{k}(\pi \mathrm{N}_{W'} E)$$
and
$$\mathscr{B}^{k-1}(\mathrm{N}_{W'} E) \ \xrightarrow{\cdot\vartheta_{W'}}\ \mathscr{B}^{k}(\pi \mathrm{N}_{W'} E, \pi\partial \mathrm{N}_{W'} E)$$
are isomorphisms. The kernel of
$$\mathscr{B}^{k-1}(\pi E, \partial\pi E) \ \xrightarrow{\cdot\overline{\vartheta_{W'}}}\ \mathscr{B}^{k}(\pi E),$$
pulled back to $\mathscr{A}(\mathrm{lk}_w\, E)$, satisfies the biased pairing property in degree $k-1$ by the first hereditary condition and Lemma 7.5. Hence, the perturbation lemma gives an isomorphism
$$\mathscr{B}^{k-1}(\pi \mathrm{N}_{W} E, \partial\pi \mathrm{N}_{W} E) \ \xrightarrow{\cdot\overline{\vartheta_{W}}}\ \mathscr{B}^{k}(\pi \mathrm{N}_{W} E).$$
Similarly, applying the second hereditary condition and Lemma 7.5, we obtain an isomorphism
$$\mathscr{B}^{k-1}(\mathrm{N}_{W} E) \ \xrightarrow{\cdot\vartheta_{W}}\ \mathscr{B}^{k}(\pi \mathrm{N}_{W} E, \pi\partial \mathrm{N}_{W} E).$$
Iterating gives the desired isomorphisms. □

9.4. **Passing to manifolds.** For manifolds, the same approach works, but we have to pass to relative homology and acyclicity, with a caveat: The suspension is not a manifold, and we have to determine the socle in terms of the homology of the manifold M.

Theorem 9.3. *Let πM be a manifold of dimension $d-1$, realized in $\Bbbk^{[d]}$, and consider the suspension $\mathrm{susp}\,\pi M$ realized in $\Bbbk^{[d+1]}$. Assume that the suspension points project to $\mathbf{0}$ in $\Bbbk^{[d]}$.*

Then the kernel of
$$\mathscr{A}^{k}(\mathrm{susp}\, M) \ \longrightarrow\ \bigoplus_{v \in M^{(0)}} \mathscr{A}^{k}(\mathrm{st}_v\, \mathrm{susp}\, M)$$
is isomorphic to
$$(H^{k-1}(M))^{\binom{[d]}{k}} \ \oplus\ (H^{k-2}(M))^{\{d+1\}\cup\binom{[d]}{k-1}}.$$

Proof. Consider north pole $\mathbf{n}$ and south pole $\mathbf{s}$. This follows from the short exact sequence
$$0 \ \longrightarrow\ \mathscr{A}^{*}(\mathrm{susp}\, M, \mathrm{st}_{\mathbf{s}}\, \mathrm{susp}\, M) \ \longrightarrow\ \mathscr{A}^{*}(\mathrm{susp}\, M) \ \longrightarrow\ \mathscr{A}^{*}(\mathrm{st}_{\mathbf{s}}\, \mathrm{susp}\, M) \ \longrightarrow\ 0$$

The left and right side are, by cone lemma, isomorphic to $\mathscr{A}^*(\pi M)$, with the left side a copy shifted by one degree. □

Hence, once again, we obtain a description of the socle of the suspension; the ring $\mathscr{B}(\operatorname{susp} M)$ is defined as the quotient by the open socle.

The proof of Theorem 1.1 for closed, orientable manifolds follows analogously from here, reducing via railways to the case of weakly deconstructible spheres. The case of orientable manifolds with boundary in Theorem 5.12 only needs minor adaptation to account for relative homology, and similarly analogous.

10. Further remarks and directions

Let us close this manuscript by first discussing the Hard Lefschetz properties.

10.1. **Hard Lefschetz and the HGL relations.** We now recall a technique of [Adi18] to establish the HGL relations and hard Lefschetz property, and in fact obtain yet another reduction of the Lefschetz property to biased pairing directly. We can restrict to orientable closed manifolds, which we close as a class under suspensions. These suspensions are chosen as in Section 9.4: If πX is realized in $\mathbb{k}^{[d]}$, then $\operatorname{susp} X$ is realized in $\mathbb{k}^{[d+1]}$ so that the suspension points project to the origin. We have $\mathscr{B}(X)$ defined as usual, the unique Gorenstein quotient of $\mathscr{A}(X)$ with the same top socle element.

To this end, assume we wish to prove the HGL relations for a pair (X, Y), where Y is a subcomplex of X a $(d-1)$-complex in the class, specifically the HGL relations for $\mathscr{K}^*(\Sigma, \Delta)$ or its annihilator. Label the two vertices of the suspension $\mathbf{n}$ and $\mathbf{s}$ (for north and south). We then argue using the following observation, which follows as Proposition 6.6. We remind ourselves that $A * B$ denotes the free join of two simplicial complexes A and B.

Lemma 10.1. *Considering* $\operatorname{susp} X$ *realized in* $\mathbb{k}^{d+1}$*, and* $k < \frac{d}{2}$*, the following two are equivalent:*

(1) The HGL relations for

$$\mathscr{K}^{k+1}(\operatorname{susp} X, \operatorname{susp}(Y) \cup \mathbf{s} * X) \quad \textit{resp.} \quad \overline{\mathscr{K}}^{k+1}(\operatorname{susp} X, \mathbf{n} * Y)$$

with respect to $x_{\mathbf{n}}$.

(2) The HGL relations for

$$\mathscr{K}^{k}(\pi X, \pi Y) \quad \textit{resp.} \quad \overline{\mathscr{K}}^{k}(\pi X, \pi Y)$$

with respect to ϑ.

Proof. Without loss of generality we have $\vartheta = x_{\mathbf{n}} - x_{\mathbf{s}}$ in $\mathscr{A}^*(\operatorname{susp} X)$. Consider then the diagram

$$\begin{tikzcd} \mathscr{A}^k(\pi X) \arrow[r, "\cdot\vartheta^{d-2k}"] \arrow[d, "\sim"] & \mathscr{A}^{d-k}(\pi X) \arrow[d, "\sim"] \\ \mathscr{A}^{k+1}(\operatorname{susp} X, \mathbf{s} * X) \arrow[r, "\cdot x_{\mathbf{n}}^{d-2k-1}"] & \mathscr{A}^{d-k}(\mathbf{n} * X) \end{tikzcd}$$

where the first vertical map is defined by the composition of cone lemmas

$$\mathscr{A}^k(\pi X) \;\cong\; \mathscr{A}^k(\mathbf{n} * X) \;\xrightarrow{\cdot x_{\mathbf{n}}}\; \mathscr{A}^{k+1}(\operatorname{susp} X, \mathbf{s} * X).$$

and the second vertical map is simply the cone lemma. An isomorphism on the top is then equivalent to an isomorphism of the bottom map, and restricting to ideals and their Poincaré duals gives the desired. □

Hence, one can inductively reduce the study to the weak Lefschetz property. The main final Theorem is then:

Theorem 10.2. *Fix an infinite field $\Bbbk$ of any characteristic. Consider a triangulated orientable compact $\Bbbk$-homology manifold M of dimension $d-1$, and the associated graded commutative face ring $\Bbbk[M]$. Then the Gorensteinification of a generic Artinian reduction $\mathscr{A}^*(M)$ of $\Bbbk[M]$ satisfies the hard Lefschetz property with respect to a generic $\ell \in \mathscr{A}^1(M)$, that is, for every $k \le d/2$, an isomorphism*

$$\mathscr{B}^k(M, \partial M) \;\xrightarrow{\cdot \ell^{d-2k}}\; \mathscr{B}^{d-k}(M).$$

10.2. **Limits of the theory: between pairings and numerical monotonicity.** [Adi18] introduced the *biased pairing property*, which leads to a proof of the Lefschetz property in arbitrary characteristic: It says that the Lefschetz property can be reduced to a nondegeneracy of Poincaré pairings when restricting to subspaces.

Papadakis and Petrotou introduced total anisotropy: the property that the Poincaré pairing does not degenerate at *any* ideal (or equivalently, at principal ideals) [PP20, APP23, APP25]. They then use the reduction of [Adi18] to prove the Lefschetz property.

That is, we have,

$$\text{anisotropy} \;\rightsquigarrow\; \text{biased pairing property} \;\rightsquigarrow\; \text{Lefschetz property}$$

with the curiosity that the biased pairing property also leads back to the Lefschetz property, in precise ways (see our Section 6).

But in the setting of the Geiriginger-Laman Theorem 5.4, and if we are permitted to mention combinatorics as well, the ubiquitous Hall's marriage theorem [Hal35], a simpler way also suffices: A numerical monotonicity property of ideals, that is implied by the Lefschetz property (see Observation 5.1) and biased pairing property like 6.3.

But Geiriginger-Laman Theorem and Hall's marriage theorem show that it suffices to have that monotonicity alone to guarantee the (weak) Lefschetz property. We arrive at the following illustrative question.

Question 10.3. *Consider a triangulated sphere Σ of even dimension $2k = d-1$, and $\mathscr{A}^*(\Sigma)$ an Artinian reduction such that for every monomial ideal (or any ideal?) $\mathscr{I}$, we have an inequality*

$$\dim \mathscr{I}^k \;\le\; \dim \mathscr{I}^{k+1}.$$

Does $\mathscr{A}^*(\Sigma)$ have the weak Lefschetz property?

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