ALL TRIANGULATIONS HAVE A COMMON STELLAR SUBDIVISION

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For Frank Hagen Lutz, in memory

Abstract. We address two longstanding open problems, one originating in PL topology, another in birational geometry. We prove the weighted version of Oda's *strong factorization conjecture* (1978), and prove that any two toric varieties whose fans have the same support have a common stellar transformation. This implies that every two PL homeomorphic polyhedra have a common stellar subdivision, which was a conjecture going back to Tietze's formulation of the Hauptvermutung in 1908, and often attributed to Alexander.

1. Introduction

Let Q be a geometric simplicial complex in $\mathbb{R}^d$, that is, a simplexwise linear (or rectilinear) embedding of an abstract simplicial complex into euclidean space of dimension d (see also Section 2 for basic notions), and let T be a triangulation of Q. Define a *stellar subdivision* at a point $z \in Q$ to be a transformation given by removing from T all faces containing z, and then adding to T the cone over the boundary of the subset removed (see Figure 3.1). We say that a triangulation T can be *obtained by stellar subdivisions* from a triangulation S if there is a finite sequence of stellar subdivisions which starts at S and ends with T. When S can be obtained by stellar subdivisions from triangulations T and T', it is called a *common stellar subdivision* (see Figure 2.1 below).

We prove the following result.

Theorem 1.1 (the weighted strong factorization theorem)**.** *Every two triangulations A, B of a geometric complex in $\mathbb{R}^d$ have a common stellar subdivision. Moreover, if both A and B have coordinates in a field extension K over $\mathbb{Q}$, then so does the common stellar subdivision.*

Over $\mathbb{Q}$, this implies the weighted version of *Oda's strong factorization conjecture* [Oda78], cf. §8.1.

Corollary 1.2. *Two toric varieties whose fans have the same support have a common stellar transformation.*

Let us provide some context. A classical result of PL topology states that every two triangulations A, B of a geometric complex in $\mathbb{R}^d$ are connected by a sequence of stellar subdivisions and their inverses, see [Ale30, AI15].

Mathematicians interested in (toric) algebraic geometry refined this further, and proved that one can restrict to blowups and blowdowns at smooth loci only. This was proved in dimension at most three in [Dan83], and in full generality in [Wło97], see also [A+02, IS10] and §8.1. The strong factorization conjecture can be stated as follows.

Conjecture 1.3 (Oda's strong factorization conjecture [Oda78])**.** *Two proper smooth toric varieties of the same dimension have a common iterated toric blowup along smooth centers.*

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Morelli claimed to have proved the strong factorization conjecture in [Mor96], which was shown incorrect in [Mat00]. In a positive direction, the conjecture was confirmed in [Mac21] for a very special class of polyhedra. In [DK11], the authors proposed an algorithmic construction, which remains unproven (cf. §8.4). Our approach is notably different, but is also constructive. Corollary 1.2 is the strongest result in the positive direction yet. The strongest previous result in the direction of the strong factorization conjecture is due to Ewald [Ewa86], who for complexes of dimension $d = 2$ proved the weighted strong factorization theorem.

It is at this point that we should note that the *combinatorial* version of the strong factorization conjecture had been an open conjecture as well, and one that in fact predates Oda's strong factorization conjecture. We obtain the following result for *combinatorial* blowups and blowdowns, that is, the case in which we are only interested in the combinatorial type of the triangulations.

Theorem 1.4 (former Alexander's conjecture)**.** *Every two PL homeomorphic simplicial complexes have combinatorially isomorphic stellar subdivisions.*

This was one of the main problems left standing in the context of the "Hauptvermutung" of Steinitz [Ste08] and Tietze [Tie08], which conjectured that homeomorphic polyhedra have a common (not necessarily stellar) subdivision, see Remark 1.5. The Hauptvermutung, as far as it concerned combinatorialising homeomorphisms, was disproven by Milnor [Mil61] for complexes and later for manifolds based on the work of Casson, Sullivan, Kirby and Siebenmann, see [Rud16].

In one of the early approaches to the subjects, Alexander [Ale30] was interested in simplifying PL homeomorphisms of polyhedral spaces (another way of saying that the spaces have a common subdivision) by observing that such homeomorphisms can be factorized into stellar moves and their inverses, thereby proving the "weak factorization" for PL homeomorphisms (see also [LN16, Pac91] and §8.3). Our theorem, which resolves a conjecture attributed to Alexander [AM03, LN16], says that every two PL homeomorphic polyhedra have a common stellar subdivision, thereby proving the strong factorization of PL homeomorphisms. In this case, we do not have a geometric meaning, but a topological one. For further context of Alexander's conjecture, see e.g. [Lik99, §4], [Hud69, p. 11].

Remark 1.5. Perhaps Alexander's conjecture should actually be attributed to Tietze [Tie08, p. 12], who formulated the *Hauptvermutung* in a way that suggests that homeomorphic complexes should have a common subdivision reached by elementary moves: *daß man durch [elementare] Unterteilung aus dem einen Schema ein Schema gewinnen kann das sich auch aus dem zweiten Schema durch Unterteilung erhalten läßt.* With the negative resolution of Hauptvermutung by Milnor, one remaining conjecture is then to resolve this for PL homeomorphic complexes, which this paper does.

It was noted by Anderson and Mnëv [AM03], that Theorem 1.4 follows from Theorem 1.1. We include a short proof in Section 7 for completeness.

2. Basic definitions and notation

Let Q be a polyhedral complex embedded in $\mathbb{R}^d$, that is, a finite set of polytopes (called *faces* of the complex) such that a face of any polytope in the set is also in the set, and such that any two polytopes in the set intersect in a common (and possibly empty) face.

We say that Q is a *triangulation*, or *geometric simplicial complex* if it is simplicial, that is, every polytope in the complex is a simplex. Some authors also call this (or think of this as) a *rectilinear embedding* of an abstract simplicial complex. The elements of a simplicial complex are also called *faces*, the maximal faces are called *facets*.

We use the same terms and notation in both geometric (realized with simplices in the Euclidean space) and topological (abstract complexes within the PL category) settings, hoping this would not lead to confusion. We use the terms "geometric triangulation", "geometric (polyhedral) complex", etc., when the distinction needs to be emphasized. However, until Section 7, we exclusively work in the geometric setting, thus avoiding this source of confusion.

Denote by $\mathcal{T}(Q)$ the set of triangulations of Q. We write $S < T$ if T is a *refinement*, or *subdivision* of S, where $S, T \in \mathcal{T}(Q)$, that is, if every simplex of T is contained in a simplex of S.

A continuous map of simplicial complexes $\varphi : T \to S$ is analogously called a *subdivision map* if for any simplex t of T, $\varphi(t)$ is contained in a simplex of S. We write $S \lhd T$ if T can be obtained from S by a sequence of stellar subdivisions. In this case we say that T is an *iterated stellar subdivision* of S. Given triangulations $S, T \in \mathcal{T}(Q)$, we will speak of a *common (iterated) stellar subdivision* to denote a triangulation $R \in \mathcal{T}(Q)$, such that $S \lhd R$ and $T \lhd R$, see Figure 2.1.

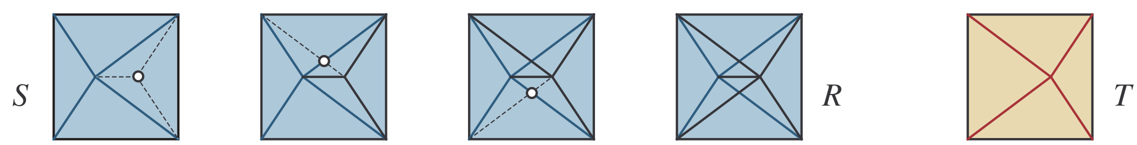

FIGURE 2.1. Triangulations S, T of a square, a common stellar subdivision R, and stellar subdivisions from S to R.

Let T be a simplicial complex and let F be a face of T. The *star* $\mathrm{st}_F T$ is the minimal simplicial subcomplex of T that contains all faces containing F. The *link* $\mathrm{lk}_F T := \partial\, \mathrm{st}_F T$ is the boundary of $\mathrm{st}_F T$ with respect to the intrinsic topology of T. We use $T - F$ to denote maximal subcomplex of T which does not contain F, also called the *antistar* of F in T in the literature. We finally remind ourselves of a useful classical lemma.

Lemma 2.1 ([Zee63, Lemma 4, p. 8])**.** *Given A and B simplicial complexes with the same underlying space, we can apply stellar subdivisions to B until it refines A. If A and B are defined over a field K, then so is the subdivision.*

3. PLANAR CASE

In this and the following two sections we are concerned *only* with geometric triangulations. In this section, we consider triangulations of a convex polygon. In the next two sections, we consider geometric complexes in higher dimensions.

Note that for 2-dimensional complexes, there are only two types of stellar subdivision, which are pictured in Figure 3.1 below: The subdivision at an edge and at a triangle.

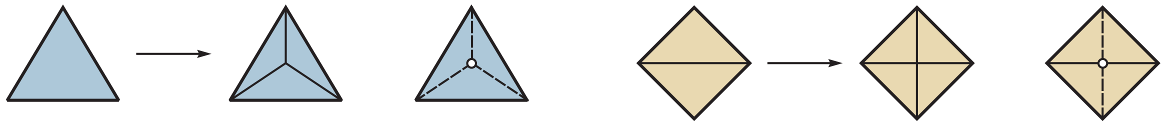

FIGURE 3.1. Two types of stellar subdivisions in the plane. The circle and dashed lines indicate the added vertices and edges.

3.1. **Triangulations of polygons.** The case of $d = 2$ is especially elegant because planar graphs are particularly simple. More importantly, the underlying idea follows the same beats as in the higher-dimensional case, so it is instructive to keep this in mind.

In this section, we present a self-contained proof of the weighted strong factorization conjecture in the plane.

Let $Q \subset \mathbb{R}^2$ be a convex polygon in the plane, and let $T \in \mathcal{T}(Q)$ be a triangulation of Q. Let $x \in Q$ be a point in the interior of a triangle (abc) in T, and let T' be a triangulation obtained from T by adding edges xa, xb and xc. Similarly, let $x \in Q$ be a point in the relative interior of an edge ab, and let T' be a triangulation obtained from T by adding edges xc for all triangles (abc) in T that have ab as an edge. A *stellar subdivision* is an operation $T \mapsto T'$ in both cases. Clearly, we then have $T < T'$.

Theorem 3.1 (strong factorization for convex polygons)**.** *Suppose T, T' are triangulations of a convex polygon Q. Then there is a triangulation $S \in \mathcal{T}(Q)$ that can be obtained by a sequence of stellar subdivisions from both T and T'.*

This theorem was proven first by [Ewa86], but the algorithm we present here generalizes to higher dimensions.

3.2. **Stellar subdivision of fins.** Let $Q \subset \mathbb{R}^2$ be a possibly non-convex polygon in the plane, that is, a disk bounded by a simple, closed, piecewise linear curve, and let V be its set of vertices. Fix a vertex $v \in V$ that we call an *anchor*.

We say that Q is *starshaped* at anchor v if $[u, v] \subset Q$ for all $u \in Q$. A geometric simplicial complex will be called starshaped if its underlying set, that is, the union of its simplices, is starshaped. Denote by ∂Q the boundary of Q. For a region $D \subset Q$, denote by $T|_D$ the restriction of triangulation T to D. Let $T \in \mathcal{T}(Q)$ be a triangulation of a starshaped polygon Q with anchor v. The *horizon* of T is the subset of T defined by points x in T such that the segment $[v, x]$ lies within T, but no line segment $[v, y]$ strictly containing $[v, x]$ is contained in T. We call T a *fin* if the horizon is a subcomplex of T (or equivalently, if the horizon is closed). The *horizon* is also denoted by $\mathrm{hr}_v T$.

We say that a polyhedron P is *compatible* with a polyhedral complex X if restricting X to the faces $X|_P$ contained in P is a subcomplex of X. In other words, if σ is a polyhedron of X whose relative interior intersects P, then σ lies in P.

We think of T as its set of triangles, and use V_T and E_T to denote the vertices and edges in T, respectively. We say that $T \in \mathcal{T}(Q)$ is a *scaled fin* (triangulation) anchored at v, if for every vertex $z \in V_T$, the triangulation T is compatible with the line segment from v to z. See, for instance, Figure 3.2, which shows a scaled fin A and a non-scaled fin B.

A scaled fin without interior vertices, that is, no vertices in addition to v and those of the horizon, is called *striped*, or simply a *stripe*.

We also consider the stripe associated to a fin T: It is the minimal stripe containing all vertices of T. An interesting case is the one when T is a scaled fin, and S is its stripe.

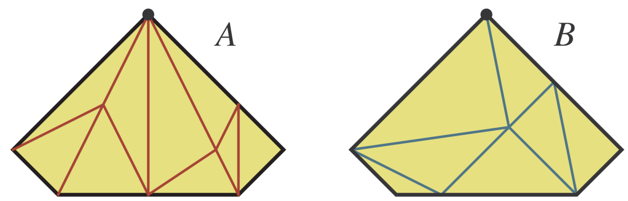

FIGURE 3.2. Two fins. A is scaled, B is not.

Lemma 3.2. *Let $v \in V$ be a vertex of the polygon $Q \subset \mathbb{R}^2$ which is starshaped at anchor v. Let $S, T \in \mathcal{T}(Q)$ be scaled fins of Q anchored at v, such that $S < T$ and that S is the stripe of T. Then $S \lhd T$.*

Proof. Use induction on the number $|V_T|$ of vertices in T. If T has no interior vertices, we have $T = S$ and the result is trivial.

In general, suppose $uw \in E_T$ is an edge of T such that the affine hull of uw contains v and the star of u coincides with the star of uw in T. We call such vertices u and the corresponding edges uw *exposed*.

By going along the boundary, it is easy to see that there exists at least one such edge uw: Consider the horizon $\mathrm{hr}_v T$ of T, which is a path, and orient the horizon $\mathrm{hr}_v T$, ordering its vertices totally. Notice that every vertex of the horizon is contained in at least one edge whose affine hull contains v. Call it the *edge towards* v.

Consider now the edges of T that are
- ◦ incident to $\mathrm{hr}_v T$ in a vertex u,
- ◦ but neither lie in it, nor
- ◦ do their affine hulls contain v.

They will come before, or after the aforementioned edge towards v, or more precisely, these edges are incident to an edge towards v preceding u in our chosen total order, or follow it.

In the former case, we denote the edge with a *down*. In the latter, we denote it with an *up*. For instance, in the first panel of Figure 3.3, the order is *up down up up* if we orient the horizon from left to right.

We have found an edge uw that is exposed if
- (i) the sequence starts with a down,
- (ii) ends with up,
- (iii) or, anywhere in the sequence, an up is followed immediately by a down.

Since, to violate (i) and (ii), it has to end with down and start with up, the third condition has to be satisfied at least once somewhere in between.

Once we found the edge, there are two cases to consider: If w is not a vertex of S, then perform a stellar subdivision in S at the point w. Then remove u from the resulting simplicial complex, and consider $T' = T|_{T-u}$ and the complex S' obtained by performing the stellar subdivision and then removing u. Then T' is a fin and S' is its stripe (with the same anchor), and T' has one less vertex than T. Apply the inductive argument to S' and T'.

If w is already a vertex of S, then removing u from T results in a smaller starshaped complex, and we induct by applying the inductive assumption to $S' = S - u$ and $T' = T - u$.

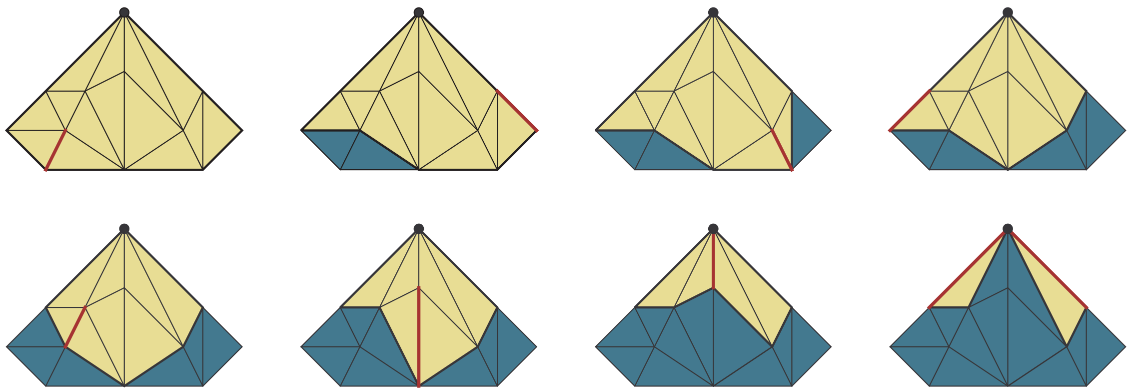

FIGURE 3.3. Shedding sequence for a scaled fin of a starshaped polygon, from left to right and top to bottom. Collapsed edges are shown in red.

Note that in the second case, the polygon Q' can become connected at v only, see the last panel of Figure 3.3. This does not affect the argument, as one can treat each component separately and proceed by induction. This completes the proof. □

Remark 3.3. The algorithm in the proof will be called the *shedding routine*: It applies a stellar subdivision, and then removes the part of the complex where the triangulations were made to coincide already, making the remaining complex smaller (in terms of the number of vertices). The sequence of shedding steps, that is, the intermediate complexes in T induced by removing the exposed vertices, is called the *shedding sequence*.

Another way to state Lemma 3.2 is then to say:

◦ Scaled fins have shedding sequences.
◦ A shedding sequence on a scaled fin gives a sequence of stellar moves on the stripe that transforms it into the fin.

3.3. **Common stellar triangulations in the plane.** We now prove Theorem 3.1. To construct a common stellar triangulation, we follow a series of steps. Start with triangulations $A, B \in \mathcal{T}(Q)$ of a convex polygon Q in the plane. Fix a vertex v of Q.

Step 1. We first show how to turn a fin into a scaled fin, by showing how to perform stellar subdivisions to make the triangulation compatible with any given line segment:

Lemma 3.4 (Weak scaling)**.** *Consider a fin C anchored at v and a collection of lines L through v containing all vertices of C. Then there is a stellar subdivision of C that is scaled, and such that for each edge of the associated stripe that is incident to v there is a line ℓ in L that strictly contains it, and conversely for each line of L there is some edge of the stripe contained in it.*

Proof. To this end, we use stellar subdivisions in C to construct a scaled fin triangulation $C' \rhd C$ that is anchored at v.

Proceed as follows. For every line $\ell \in L$, the intersection of ℓ with the underlying polygon is a line segment from v to a point w. We will add such line segments one by one in any order, until the desired scaled fin is obtained.

To add vw, note that vw intersects the existing edges $ab \in E_C$ transversally, unless ab is already compatible with the triangulation. Make a stellar subdivision at points of

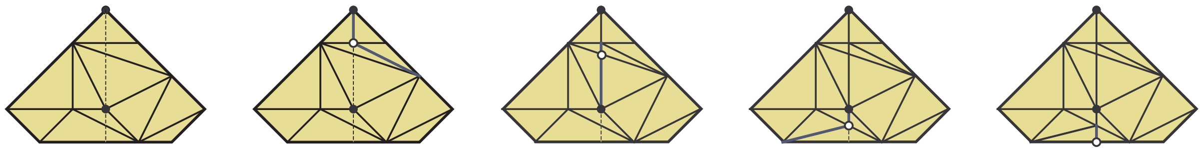

FIGURE 3.4. Adding a dotted line in Step 1 using stellar subdivisions.

intersection $vw \cap ab$ in the order from v to w. At each subdivision, the first added edge is along vw while another may diverge. The last of the intervals to be added is along vw adjacent to w, see Figure 3.4. □

Apply this lemma to A, thereby turning it into a scaled fin A'.

Step 2. Use stellar subdivisions in B to construct a fin triangulation $B' \triangleright B$ that is anchored at v and refines A', i.e., $A' < B'$, using Lemma 2.1.

Step 3. Compute a *pulling sequence* of A', that is, a sequence

$$A' = A'_0 \supset A'_1 \supset A'_2 \supset \ldots \supset A'_\ell = v$$

of intermediate subcomplexes that are fins with respect to v, and such that D_i, the minimal subcomplex of A' containing $A'_{i-1} \setminus A'_i$, is a stripe with respect to a vertex w_i in A'_i, not in $\mathrm{hr}_v A'_i$, and $w_i \neq w_j$ provided $i \neq j$. Shedding sequences are pulling sequences, but the other direction does not hold, and we need that extra bit of flexibility in Step 4.2 later. For simplicity, we shall also impose that $\mathrm{hr}_{w_i} D'_i$ is an *induced subcomplex* of A', that is, a simplex of A' is in the former if and only if all its vertices are.

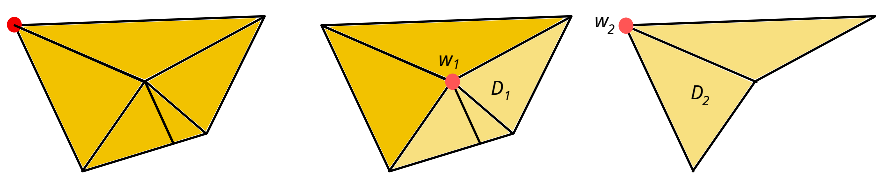

FIGURE 3.5. A pulling sequence for a complex anchored at a vertex marked in the upper left corner. The pulling sequence decomposes the complex A into a sequence of stripes, each of which is refined by B. We apply Lemma 3.2 on each of these stripes separately.

This is easy to do, using the following observation:

Lemma 3.5. *Every shedding sequence is a pulling sequence.*

Proof. As in the proof of Lemma 3.2, we record the sequence of edges incident to the horizon but not contained in it, and whose affine span does not contain the anchor (recall, those were the *exposed edges*): Find either a down in the beginning, or a down immediately following an up, or an up at the end; the corresponding subcomplex D_1 is given as the closed neighborhood of the corresponding vertex. Induct, and obtain the pulling order. □

Step 4: Loop. Assume now $A'|_{D_j} = B'|_{D_j}$ for all $j < i$. We now describe a sequence of stellar subdivisions in $A'|_{D_i} = B'|_{D_i}$ that arrange it so that $A'|_{D_j} = B'|_{D_j}$ for all $j \leq i$, and such that the sequence of refinements $A'_{i+1} \supset \ldots \supset A'_\ell$ remains a pulling sequence.

Step 4.1 Since each region D_i is starshaped at w_i, we have that A' restricted to D_i is a stripe anchored at w_i, and that B' restricted to D_i is a fin anchored at w_i. Using Lemma 3.4, we can perform stellar subdivisions in both $A'|_{D_i}$ and $B'|_{D_i}$, so that $B'|_{D_i}$ is scaled with respect to w_i, and that $A'|_{D_i}$ is its stripe, by introducing line segments ending at w_i for every vertex of $B'|_{D_i}$ and $A'|_{D_i}$. Finally, stellar subdivisions that affect the triangulations in D_j, $j < i$, coincide in both B' and A'. For clarity and for later use in the higher dimensional case, let us extract the underpinning lemma we use here:

Lemma 3.6. *Consider a simplicial complex Δ, and an induced subcomplex Γ of Δ. Consider two sequences of stellar subdivisions of faces in Γ, so that we reach subdivisions Γ' and Γ'' of Γ. Apply the same stellar subdivisions to $\Delta \supset \Gamma$, thereby obtaining two a priori different complexes Δ' and Δ''. If Γ' coincides with Γ'', then also the associated stellar subdivisions Δ' and Δ'' coincide.*

Proof. Since any face F of Δ intersects Γ in a unique maximal face G because the latter is induced in the former, it suffices to consider the case when $\Delta = G * H$ and $\Gamma = G$, that is, Δ is the free join[1] of G with another simplex H. Then no matter what stellar subdivision order is chosen, if G' is the subdivision of G, the associated subdivision of Δ is $\Delta' = G' * H$. $\square$

Step 4.2 Apply Lemma 3.2 to $A'|_{D_i}$ and $B'|_{D_i}$ to obtain a stellar subdivision of A' which coincides with B' on D_i. Once again, this affects $A'|_{D_j}$ and $B'|_{D_j}$ in the same way for all $j < i$, and hence does not break the previous achievement that $A'|_{D_j} = B'|_{D_j}$ for all $j < i$ by Lemma 3.6. Hence, we have performed the stellar subdivisions guaranteeing $A'|_{D_j} = B'|_{D_j}$ for all $j \leq i$.

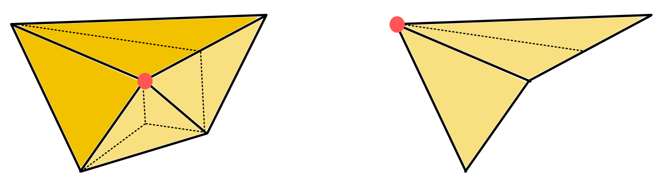

FIGURE 3.6. Stellar subdivisions in D_i may affect some D_k, $k > i$. But A, restricted to these sets D_k, remains a stripe all the same, as seen in the right figure where A restricted to D_2 has one additional vertex obtained by a stellar subdivision in its horizon, preserving the pulling.

It remains to discuss how the triangulations $A'|_{D_k}$ and $B'|_{D_k}$ are affected for $k > i$. We do not actually care about the latter for the algorithm, but the former we need to spend a minute on: We need to argue that the pulling sequence is not affected. While the common stellar subdivision of D_i is constructed by the shedding routine of Lemma 3.2, some stellar subdivisions can be made for vertices $z \in D_i \cap A'_i$ on the common boundary of both regions D_i and A'_i. In these cases, new edges zw_k are added to A', for $k > i$. Clearly, the restriction of A' to the corresponding D_k remains a striped fin with respect to w_k, and since subdivisions preserve the property of being induced, the remainder of the pulling

[1]Recall that the *free join* of two simplicial complexes A and B is the simplicial complex consisting of the pairs (a, b) for $a \in A$ and $b \in B$. Geometrically, this can be realized by embedding the complexes in skew subspaces and considering all the simplices formed by taking convex hulls of simplices in A and B.

remains unaffected. In other words, the remainder of the sequence $A'_{i+1} \supset \ldots \supset A'_\ell = v$ remains a pulling for A'_{i+1} after the subdivisions made, see Figure 3.6.

Step 4: End of loop. Proceed by induction on i and repeat Step 4 until the pulling of A' is complete, obtaining the desired common stellar subdivision. At the end, we obtain $A \lhd A' \lhd B'$ and $B \lhd B'$, as desired. This proves Theorem 3.1. □

4. General algorithm, preparation

Let us reintroduce some of the notions of the previous section in a more general form.

4.1. **Anchors, fins, stripes and scales.** A polyhedral complex T is *starshaped* if its underlying set is starshaped with respect to a point v, that is, for every point x in the set, the line segment $[x, v]$ from x to v is in the set. The point v is called the *anchor*. Unlike the previous section, we do not require this point to be a vertex, or even a boundary point.

We continue with our starshaped polyhedral complex T with anchor v. Once again, the *horizon* of T is the subset of T defined by points x in T such that the segment $[v, x]$ lies within T, but no line segment $[v, y]$ strictly containing $[v, x]$ is contained in T. We call T a *fin* if, in addition to being starshaped, the horizon is a subcomplex of T (or equivalently, if the horizon is closed.)

We denote this horizon by $\mathrm{hr}_v T$. The fin T is called *stripe* if it coincides with $v * \mathrm{hr}_v T$, the cone over $\mathrm{hr}_v T$ with apex v, and the latter complex is also called the *stripe* of T. In other words, $v * \mathrm{hr}_v T = \{\mathrm{conv}(\{v\} \cup F) : F \in \mathrm{hr}_v T\} \cup \mathrm{hr}_v T$.

A related, and central notion is that of *scaled fins*. A fin T is *scaled* if the radial projection $\rho_v : T \setminus \{v\} \to \mathrm{hr}_v T$ maps every simplex of T that is *not* v, to a simplex of $\mathrm{hr}_v T$, that is, the image under the radial projection of a simplex in T (that is not v) in $\mathrm{hr}_v T$ is a simplex in the latter.

4.2. **Sheddings and presheddings.** Continuing with a fin T, we say a vertex w in $\mathrm{hr}_v T$ is *exposed* if there is an edge $E = E_w$ of T *not* in $\mathrm{hr}_v T$ such that $\mathrm{st}_E T = \mathrm{st}_w T$, and such that the edge E is contained in the convex hull of v and w. We say in this case that T has a *shedding* to $T - w$, the maximal subcomplex of T not containing w. The vertex $w' = E - w$ is called the *shedding vertex*. We say T is *sheddable* if there is a sequence of sheddings such that the T is reduced to a vertex. A useful example is the following.

Example 4.1. If T is a scaled fin whose stripe is a simplex, then it has a shedding.

The following observation is useful:

Proposition 4.2. *Consider a sheddable scaled fin T. Then T is a stellar subdivision of its stripe.*

Indeed, observe that the dual graph of the triangulation is a path; one of its leaves is incident to v, and unless it consists of a single simplex, the other one contains a vertex that is in no other facet of the triangulation. This vertex is exposed. Proceed by induction.

Proof. Consider the shedding vertices in their natural order, and perform stellar subdivisions at these vertices in precisely that order. This transforms the stripe into the fin. □

This is but part of Lemma 3.2, as we discussed in Remark 3.3. The difference in dimension 3 and up is that sheddings do not always exist.

We therefore now introduce the notion of *preshedding*. We once again consider a fin T with anchor v. We say a face F of $\mathrm{hr}_v T$ is *exposed* if there is a unique face F' containing F

such that $\mathrm{st}_F T = \mathrm{st}_{F'} T$, and whose affine hull contains v. We then say T has a *presheddding* to $T - F$, the maximal subcomplex of T not containing F, if this complex is a fin with respect to v as well.

The face F' is also called the *preshedding face*, the face F the *preshedding loss*, and T is *presheddable* if it can be reduced to a single simplex using preshedding steps.

We have the following fact:

Lemma 4.3. *Consider a triangulation S of a polytope P in $\mathbb{R}^d$. Then for any point p in P there exists an iterated stellar subdivision of S that is presheddable with respect to the anchor p.*

Proof. The proof is classical, and exploits the fact that presheddings are closely related, though not quite the same, as line shellings, see [Zie95, §8.2].

Recall that after sufficiently many stellar subdivisions, the triangulation S becomes *regular* [AI15], i.e., there is a concave piecewise linear function whose domains of linearity are exactly the faces of the subdivision S' of S. In other words, we can lift S' to be the boundary of a convex polyhedron in $\mathbb{R}^d \times \mathbb{R}$. Choose a generic such lift with respect to p.

We now use the following Bruggesser–Mani trick from [BM71]. Pick the unique point p' on this lifted surface that projects to p, and move outwards from it along a half-line ℓ to infinity away from the surface in the same direction as the lift (in the apt imagery of [Zie95, §8.2], "launch a rocket upwards").

Record the order of hyperplanes spanned by the facets of S' encountered along ℓ. If p is generic, this gives a total order on the facets of S', and by reversing it, the preshedding order. In this case, all preshedding faces are facets, and a preshedding is just a shelling in the classical sense.

If p is not generic, there may be several hyperplanes, that is, facets of S' encountered at the same time as the rocket flies up. Since the lift is generic with respect to p, however, the intersection of these facets encountered at the same time then becomes the desired preshedding face. Indeed, when projected back $\mathbb{R}^d$, it is straightforward that all intermediate complexes are fins with respect to anchor p and that the steps are individually presheddings. □

Remark 4.4. Lemma 4.3 also follows from [AB17, Thm A], which states that the triangulation S of P becomes shellable after two barycentric subdivisions. Note that a barycentric subdivision is a composition of stellar subdivisions: first in all simplices of maximal dimension, then in all simplices of codimension one dimension, etc. In fact, it follows from the proof in [AB17], that the resulting shellable triangulation T remains a fin with respect to anchor p throughout the preshedding. Since this result is not explicitly stated (and considerably more technical), we include a simple alternative proof above. However, if one is interested in minimizing the number of stellar subdivisions, the approach of Adiprasito and Benedetti is substantially more efficient.

5. The scaling algorithm

In this section we present a crucial proposition on the way to the weighted strong factorization conjecture: an algorithm that scales a fin. It is one of the key issues that is more difficult in higher dimensions compared to the planar case, though the algorithm also works in the planar case. It will take some time to prove.

Proposition 5.1. *Let $T \subset \mathbb{R}^d$ be a triangulation of a d-polytope P, and let v be an interior point of P. Then T has an iterated stellar subdivision T' that is also a scaled fin anchored at v. Moreover, we can choose T' so that it has a shedding with respect to that anchor.*

We will provide some subroutines before we can give the proof.

5.1. The star-gazing subroutine, and halffins.

Lemma 5.2 (Refining scalings and star-gazing subroutines)**.** *If T is a scaled fin with anchor v, $H = \mathrm{hr}_v T$ its horizon, and H' is any stellar subdivision of H, then some stellar scaled subdivision T' of T has horizon H'. Moreover, if T has a shedding, then T' can be chosen to have a shedding as well.*

Proof. We may assume that H' is obtained from H by a single stellar subdivision, introducing a vertex p to H. Consider the line segment $[v, p]$. Consider the faces of T it intersects transversally, that is, in a set of dimension 0, and order them from v to p. Perform stellar subdivisions at these points in this order.

To finalize, we need to make clear that this process preserves sheddability: To understand this, we trace the shedding of T' through the lens of the shedding of T.

Consider the first exposed vertex w of the shedding order for T. If $\mathrm{st}_w T$ does not intersect $[v, p]$ in its interior, there is nothing to be done and we leave the shedding order as is, removing the star of w.

If not, then p is exposed in T'. Modify the shedding order by removing the star of p in T' first, then remove the now exposed vertex w. Repeat this reasoning with $T - w$ and the restriction $T'|_{T-w}$ of T' to the support of $T - w$. □

The proof above defines a subdivision which we call the *star-gazing subroutine* of T at the ray $\overrightarrow{vp}$, the ray from v through p to infinity.

We need to allow for more general intermediate objects. We call T a *halffin* with anchor v if there is a fin S with anchor v such that $T \cup S$ is a fin with anchor v, and $T \cap S$ is a subcomplex of $\mathrm{hr}_v S$. We call $\mathrm{hr}_v T := T \cap \mathrm{hr}_v(T \cup S)$ the *horizon* of T. We moreover consider the *shore*, that is, the closure $\mathrm{sr}_v T$ of all points s in T such that the open segment $(v, s) = [v, s] \setminus \{v, s\}$ is disjoint from T, see Figure 5.1.

A *shedding step* of T is a shedding step of $T \cup S$ not in S. We call a halffin T *sheddable* if the shore and horizon coincide, or there exists an exposed vertex in the horizon not in S whose removal results in a sheddable complex.

A halffin with anchor v comes with two radial projections: ρ_v is the radial projection to the horizon, and, if v is disjoint from T, a radial projection ρ^v to the shore.

Example 5.3. Any simplex in $\mathbb{R}^d$ is a halffin with respect to any choice of anchor v.

We summarize the notions of this section into a subroutine, the star-gazing subroutine:

Star-gazing subroutine. Input: *A halffin T with anchor v in $\mathbb{R}^d$, and a ray ρ emanating from v.*

Output: *A stellar subdivision of T that is compatible with ρ, that is, every face of the subdivision that intersects ρ in its relative interior is contained in ρ.*

We already described this in the proof of Lemma 5.2: We order the points of ρ totally from v to infinity, and consider the transversal intersections of ρ with faces of T in the induced order. Perform stellar subdivisions at these points in T in this order. ■

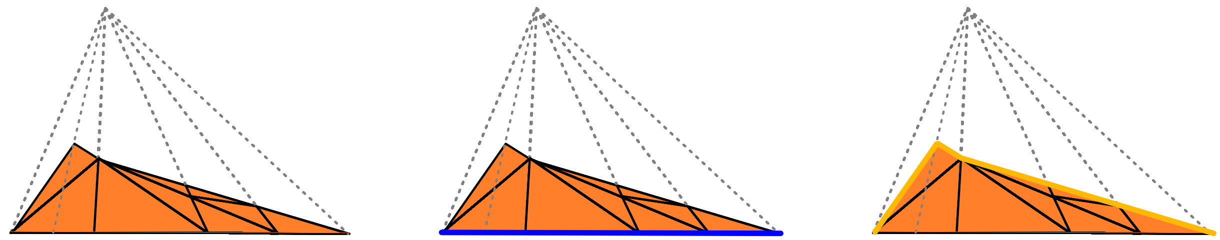

FIGURE 5.1. An instance of a halffin, the associated blue horizon at the bottom, and the associated sandy shore covering the top.

Let us record immediate properties of the algorithm that we already observed in the proof of Lemma 5.2.

Lemma 5.4 (Concerning the star-gazing subroutine)**.** *Let T, v, ρ, be an input of the star-gazing subroutine. All terms are with respect to the anchor v.*
a) If the input of the star-gazing subroutine is sheddable, then so is the output.
b) If the input is scaled, then so is the output.

5.2. **Reanchoring.** We need another subroutine for the scaling algorithm, that is, the algorithm that proves Proposition 5.1. For this, we start with the notion of a *conic halffin* with respect to an anchor v:

A *conic halffin* is a halffin T that triangulates a convex polytope Q disjoint from v with a distinguished point $q \in \partial Q$ such that

- ∘ T is a stripe with respect to q,
- ∘ q is in the shore of T with respect to v,
- ∘ the horizon of T with respect to q is the horizon with respect to v,
- ∘ the radial projection ρ^v with center v that sends the horizon to the shore is a subdivision map, that is, the image $\rho^v(\sigma)$ of any face σ of the horizon lies in some face of the shore and finally
- ∘ the horizon is scaled and sheddable with respect to the image $\rho_v(q)$ of q.

This wordy setup is followed with a neat algorithm, summarized in the following simple philosophy:

We have an object that is not only scaled, but striped with respect to an anchor q. We want it to be scaled with respect to a different anchor v. See also Figure 5.2.

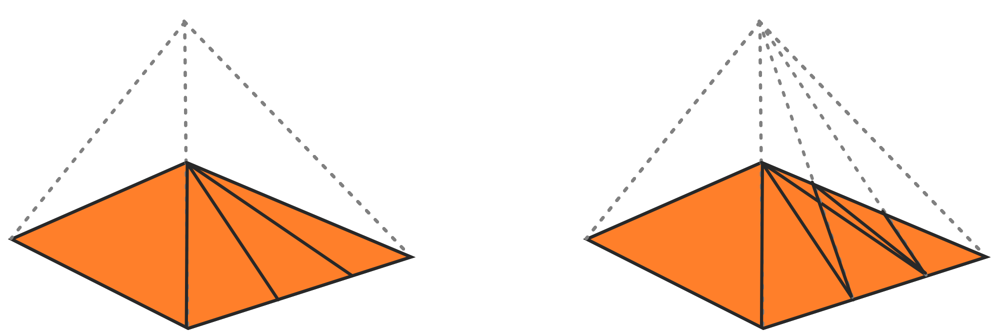

FIGURE 5.2. Reanchoring a conic halffin from one anchor to another is a way to transfer the anchor of a scaling.

Reanchoring subroutine.

Input: *A conic halffin T with anchor v.*

Output: *A stellar subdivision of T that is scaled and sheddable with respect to v that leaves the horizon invariant.*

The description of the algorithm is actually much more straightforward than its setup: By assumption, the horizon H is sheddable with respect to $\rho_v(q)$. Record the shedding vertices, that is, the vertices that are removed in the shedding sequence of H, in their order $s_1, s_2, \cdots$, ending with $s_k = \rho_v(q)$.

Now, perform star-gazing subroutines of T with respect to the rays $\overrightarrow{vs_i}$, in that order starting from s_1. ■

We single out a lemma that records the result of this algorithm.

Lemma 5.5. *With a valid input, the output of the reanchoring subroutine returns a scaled and sheddable halffin with respect to v.*

Proof. To see that the complex is scaled, it suffices to see that it is sheddable. The shedding is found explicitly as well: We consider the segments $[v, s_i]$, but now in reverse order, starting with $[v, s_k = \rho_v(q)]$. The first exposed vertex is $s_k = \rho_v(q)$, then work our way inwards along $[v, s_k]$ towards v. Once only q is left, continue with s_{k-1} and work inwards along $[v, s_{k-1}]$ until we reach the shore, then continue with $[v, s_{k-2}]$ and so on. □

5.3. **Stripes and scales: Description of the scaling algorithm.** We now present the algorithm that proves Proposition 5.1.

Scaling algorithm.

Input: *A triangulation T of a d-polytope P and an interior point v.*

Output: *A scaled, sheddable stellar subdivision of T with anchor v.*

There is a delicate aspect of this algorithm, in that it uses induction on the dimension. We therefore use the algorithm itself as a subroutine, assuming we have described it for polytopes of dimension $d-1$. For 0-dimensional polytopes, that is, points, it is trivial, of course.

Step 1. As observed in Lemma 4.3, we can assume that T is presheddable.

Step 2: Loop. We iterate over preshedding steps indexed by parameter i. Let F_i denote the complex of the first i preshedding faces, and let T_i denote the union of their stars, and notice that it is a halffin. T_0 is the empty complex.

Assume that we already described, by induction on i, the stellar subdivisions to make T_i scaled and sheddable, turning it into a new complex T_i'.

We now find a new series of subdivision steps to make T_{i+1} scaled and sheddable as follows.

Step 2.1 Let F_{i+1} denote the next preshedding face, and $\mathrm{st}_{F_{i+1}}T$ its star. If F_{i+1} contains v, set $q_{i+1} = v$ and skip to the next step.

Otherwise, there is a unique point q_{i+1} in the shore of F_{i+1} such that, after stellar subdivision at q_{i+1}, the radial projection ρ_v from shore to horizon is a subdivision map. Specifically, if f_{i+1} is the minimal interior face of the horizon of F_{i+1} with respect to v, and c_{i+1} is the minimal face of the shore, then q_{i+1} is the unique intersection point of c_{i+1} with the image of f_{i+1} under the projection ρ^v from horizon to shore.

Perform the stellar subdivision of T_{i+1} at q_{i+1}.

Step 2.2 Perform the stellar subdivisions of the induction step, that is, the stellar subdivisions to make T_i scaled and sheddable (thereby transforming it into T_i'), but apply them to T_{i+1}. Of course, this can affect $\mathrm{st}_{F_{i+1}}T$ as well, so that the resulting subdivision of T_{i+1} is

$$\widetilde{T}_{i+1} \;=\; \Sigma_{i+1} \;\cup\; T_i',$$

where Σ_{i+1} is the subdivision $\mathrm{st}_{F_{i+1}}T$ induced by the stellar subdivision at q_{i+1}, followed by the stellar subdivision of T_i to T_i' that only stellarly subdivide at points in the horizon. Hence Σ_{i+1} is a stripe with respect to q_{i+1}, and a halffin with respect to v.

Step 2.3 The horizon $\mathrm{hr}_v\Sigma_{i+1}$ coincides with the horizon $\mathrm{hr}_{q_{i+1}}\Sigma_{i+1}$. Let ρ_v be the radial projection from the shore to the horizon of Σ_{i+1} with respect to v, and let $\widetilde{q}_{i+1} = \rho_v(q_{i+1})$ be the image of q_{i+1} in the horizon if $v \neq q_{i+1}$, and some arbitrary interior point of $\mathrm{hr}_v\Sigma_{i+1}$ otherwise.

Using the scaling algorithm for $(d-1)$-dimensional polytopes, we can find a sequence of stellar subdivisions of $\Sigma_{i+1} \cap T_i'$ at points $s_1, s_2, \cdots$ to make the latter a scaled, sheddable fin with respect to the anchor $\widetilde{q}_{i+1}$.

We realize those subdivisions in $\mathrm{hr}_v\Sigma_{i+1} \cup T_i'$ by performing star-gazing subroutines at the rays $\overrightarrow{vs_1}$ (the ray from v through s_1 to infinity), then $\overrightarrow{vs_2}$ and so on.

Performing these subdivisions in $\Sigma_{i+1} \cup T_i'$ performs stellar subdivisions in Σ_{i+1} only in the horizon. The resulting subdivision T_i'' of T_i is still scaled and sheddable by Lemma 5.4. The resulting subdivision Σ_{i+1}' of Σ_{i+1} is a conic halffin if $v \neq q_{i+1}$, and scaled sheddable with anchor q_{i+1} in any case.

If $v = q_{i+1}$, we can set

$$T_{i+1}' \;=\; \Sigma_{i+1}' \;\cup\; T_{i+1}'',$$

which is scaled and sheddable as a halffin, and go back to the first step of the loop with the next preshedding step of T.

Step 2.4 Otherwise, perform the reanchoring subroutine at Σ_{i+1}', resulting in a scaled, sheddable subdivision Σ_{i+1}'' with anchor v. We set

$$T_{i+1}' \;=\; \Sigma_{i+1}'' \;\cup\; T_{i+1}''.$$

It is scaled and sheddable, and we restart the loop.

Step 2: End of loop. Finish once the preshedding steps of T have been exhausted. The result is scaled and sheddable. ■

6. General case of the weighted strong factorization theorem

We can now finalize the proof of the weighted strong factorization theorem (Theorem 1.1). We provide the following algorithm.

Weighted strong factorization algorithm.

Input: *A is a simplicial complex of dimension d, and B is a refinement of A.*

Output: *A common iterated stellar subdivision of A and B.*

Step 1. Order the facets F_i in any order, and let v_i be interior points in each F_i (arbitrarily chosen). Perform stellar subdivisions in A at the points v_i. Restricted to any facet F of A, A' is a stripe $A'|_F$.

Step 2: Loop. Pick the largest i such that when restricted to the complex T_i of the first $i-1$ facets in the imposed order, the triangulations A' and B' coincide.

Step 2.1 Consider the facet F_i. Restricted to this facet, A' is a stripe $A'|_{F_i}$. Use the scaling algorithm to make $B'|_{F_i}$ a sheddable, scaled fin with anchor v_i.

Consider now the horizons $\mathrm{hr}_{v_i} A'|_{F_i}$ and $\mathrm{hr}_{v_i} B'|_{F_i}$. By induction on the dimension, they have a common stellar subdivision. Hence, we can apply Lemma 5.2 and apply stellar subdivisions until $A'|_{F_i}$ is the stripe of the scaled and sheddable fin $B'|_{F_i}$.

Note that this may affect $A'|_{F_k}$ for $k > i$, but the stellar subdivisions occur only in the boundary of these subcomplexes, so that they remain stripes with the same anchors v_k. Further, any stellar subdivisions affecting the restrictions to F_j, $j < i$, affect A' and B' in the same way by Lemma 3.6, so that we do not undo the progress we already made. Indeed, focus on a facet F_j, $j < i$. The triangulations of A' resp. B', restricted to $F_j \cap F_i$, are induced subcomplexes of A' resp. B' because $F_j \cap F_i$ is a convex polytope compatible with the triangulations.

Step 2.2 Use Proposition 4.2, applied to $B'|_{F_i}$, to find a stellar subdivision of $A'|_{F_i}$ that coincides with B'. Now, the triangulations A' and B' coincide on T_i. Note that the boundary of $A'|_{F_i}$ is not affected.

Step 2: End of loop. Repeat the loop until all F_i's have been dealt with. The result is a common stellar subdivision. ■

Proof of Theorem 1.1. Recall that we may assume that B refines A by Lemma 2.1. Apply the weighted strong factorization algorithm. The weighted strong factorization algorithm gives the first part of the theorem. For the second part, note that if all vertices have coordinates over K, then so do all hyperplanes and their intersections. This implies that the whole construction is defined over K, as desired. □

7. Proof of Alexander's conjecture

It was shown in [AM03] that Theorem 1.4 follows from Theorem 1.1. We include a short proof for completeness.

Proof. Let A and B be two simplicial complexes, and let $\varphi : A \to B$ be a PL homeomorphism. Observe that by pulling back the triangulation of B to A, we can find a subdivision A' of A such that $\varphi : A' \to B$ is linear on every face of A'.

Observe now that if A'' is a stellar subdivision of A that refines A', then $\varphi : A'' \to B$ is linear as well. Hence, we can think of B as a geometric simplicial complex, and A'' as a geometric simplicial complex subdividing it. Apply the weighted strong factorization theorem (Theorem 1.1) to obtain a common stellar subdivision of A'' and B, and therefore of A and B. □

8. Final remarks and open problems

8.1. **Unweighted strong factorization conjecture.** Now that the weighted strong factorization conjecture is settled (Corollary 1.2), it is natural to ask about the *unweighted strong factorization conjecture* [Oda78]. This conjecture concerns lattice fans in an ambient lattice Λ.

Consider two simplicial, unimodular fans with the same support. Here by *unimodular* we mean that the fan is generated by lattice vectors, and that the lattice points in each defining ray ρ of a simplicial cone σ span the sublattice generated by $(\operatorname{span} \sigma) \cap \Lambda$.

The difference between smooth and ordinary stellar subdivisions occurs when introducing a new vertex z (or, in the language of cones, a new ray): Whereas in a stellar subdivision, we are allowed to choose the location freely in the relative interior of the subdivided face, the *smooth* stellar subdivision only allows to introduce the lattice point $z = \sum e_\rho$, where the summation is over the rays ρ defining σ, and e_ρ is the lattice point in ρ generating $\rho \cap \Lambda$.

Question 8.1. *Consider two unimodular fans of the same support. Are there two common iterated stellar subdivisions at* smooth *centers?*

The algorithm as presented does not give this result. It is easy to see that the scaling algorithm can be modified to work with respect to the restrictions to smooth subdivisions. Unfortunately, to move from a sheddable scaled triangulation to a common *smooth* stellar subdivision seems difficult: The final algorithm introduces stellar subdivisions at the shedding points, and these shedding points usually will not correspond to the smooth subdivision.

8.2. **Toroidalization and general varieties.** It is natural to ask whether Oda's program for toric varieties extends to general varieties connected by birational maps. This is an open problem, and subject of the *toroidalization conjecture* [AMR99]. Without getting technical, the question is whether a birational morphism of varieties can be turned, after blowups at smooth centers, into a morphism of toric varieties. Thanks to the work of Cutkosky [Cut07], this is settled for varieties up to dimension 3.

8.3. **Distance between topological triangulations.** For PL manifolds in dimensions $d \geq 4$, the problem of homeomorphism is undecidable [Mar58]. This implies that the number of stellar subdivisions needed in Theorem 1.4 is not computable. We refer to [AFW15, Lac22] for detailed surveys of decidability and complexity of the homeomorphism and related problems.

However, once a PL homeomorphism between the two polyhedra is given (such as, when the two polyhedra subdivide the same subset of euclidean space as geometric simplicial complexes), then our algorithm becomes constructive, and one can in particular ask for the smallest common stellar refinement. Our algorithm is not very optimized however, and we sometimes sacrificed efficiency for simplicity (see for instance Remark 4.4.) We ask the following question:

Question 8.2. *Given two triangulations of the same polytope P, what is the smallest common stellar refinement of both?*

8.4. **Da Silva and Karu's algorithm.** Note that our choices of stellar subdivisions are asymmetric with respect to triangulations (that is, the two triangulations we wish to find a common subdivision of play a different role, and an input of pair (A, B) may give a different result from the input (B, A)) and use a delicate ordering given by the shedding routine. In [DK11], the authors proposed an algorithm for common stellar subdivision and conjectured that it works in finite time. It would be interesting to see if our proof of Theorem 1.1 helps to resolve the conjecture. Note that this would simultaneously give a positive answer to Question 8.1, since the algorithm of Da Silva and Karu uses only smooth stellar subdivisions.

8.5. **Dissections.** For dissections of polyhedra, there is a natural notion of *elementary dissection* which consists of dividing a simplex into two. Motivated by applications to scissors congruence, Sah claimed in [Sah79, Lemma 2.2] without a proof, that every two dissections of a geometric complex have a common dissection obtained as composition of elementary dissections. It would be interesting to see if the approach in this paper can be extended to prove this result.

Note that both stellar subdivisions and bistellar flips are compositions of elementary dissections and their inverses; these are called *elementary moves*. Ludwig and Reitzner proved in [LR06] that all dissections of a geometric complex are connected by elementary moves. For convex polygons in the plane, see a self-contained presentation of the proof in [Pak10, §17.5]. We refer to [LR06] also for an overview of the previous literature, and for applications to valuations.

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