STOKER'S THEOREM FROM POLYTOPAL TENSEGRITIES

MARTIN WINTER

Abstract. This document gives a short derivation of Stoker's theorem starting from the results of [3]. It is distilled from a document produced by Arseniy Akopyan together with LLM.

1. The statement

Stoker's conjecture (now a theorem [1, 2]) states the following:

Theorem 1. *Let $P, P' \subset \mathbb{R}^d$ be two combinatorially equivalent polytopes. Let $n_i, n_i' \in \mathbb{R}^d$ be the outwards pointing unit normal vectors of the i-th facet in P and P' respectively. Suppose that whenever two facets i and j are adjacent, then their dihedral angles satisfy*

$$\pi - \arccos\langle n_i, n_j\rangle \;=:\; \theta_{ij} \;\leq\; \theta'_{ij} \;:=\; \pi - \arccos\langle n_i', n_j'\rangle.$$

Then $n_i = Tn_i'$ for all facets i and some fixed orthogonal transformation T.

2. The proof

We may assume $0 \in \mathrm{int}(P)$ and $0 \in \mathrm{int}(P')$. Let $h_i, h_i' > 0$ the heights of the i-th facet over the origin in P and P' respectively. The vertices of the polar dual P° are given by $p_i = n_i/h_i$. Let $G = (V, E)$ be the graph of P°.

Now consider the points $p_i' := n_i'/h_i$. Notice that the denominator is h_i, not h_i', and the p_i' are therefore not the vertices of the polar dual of P', and in fact, not necessarily the vertices of any realization of P°.

With this we have $\|p_i\| = \|p_i'\|$, and from the increasing dihedral angles also $\|p_i - p_j\| \geq \|p_i' - p_j'\|$ for all edges $ij \in E$. We would like to invoke the following specialized version of [3, Lemma 4.3]:

Lemma 2. *Let P° have vertices p_i and graph $G = (V, E)$. If $p' : V \to \mathbb{R}^d$ satisfies*

- (*i*) *$\|p_i\| = \|p_i'\|$ for all $i \in V$,*
- (*ii*) *$\|p_i - p_j\| \geq \|p_i' - p_j'\|$ for all $ij \in E$,*
- (*iii*) *$0 \in \phi(\mathrm{int}(P^\circ))$, where ϕ is the Wachspress map $\phi : P^\circ \to \mathrm{conv}\{p_i' \mid i \in V\}$,*

then $p_i = Tp_i'$ for some orthogonal transformation T.

The missing condition (*iii*) is technical and we deal with it momentarily. Its interpretation is that the origin is "properly surrounded" by the points p_i'. If we assume it, the lemma clearly gives $n_i = Tn_i'$ for all facets and proves Theorem 1.

We do not give a full definition of the Wachspress map but merely recall that it is a continuous map $\phi\colon P^\circ \to \mathrm{conv}\{p_i' \mid i \in V\}$ with the following property:

($*$) if $f \subseteq P^\circ$ is a face then $\phi(f) \subseteq \mathrm{conv}\{p_i' \mid p_i \in f\}$.

Date: September 3, 2026.

Let's first look at the case $h_i = h'_i$: then $p'_i = n'_i/h'_i$ would be the vertices of the polar dual P'°, and hence the vertices of a realization of P°. Then (iii) follows from a simple degree argument given in [3, Section 4.4]: from $(*)$ follows that ϕ maps each face of P° to its corresponding face in P'°, and hence restricts to a degree-1 map $\partial P^\circ \to \partial P'^\circ$ between the boundaries. Since $0 \in \mathrm{int}(P'^\circ)$, Brouwer's fixed point theorem yields $0 \in \phi(\mathrm{int}(P^\circ))$.

It remains to argue that this stays true for $h_i \neq h'_i$. For this, consider the linear homotopy

$$p'_i(t) := \Big(\frac{t}{h_i} + \frac{1-t}{h'_i}\Big) n'_i, \quad t \in [0,1]$$

moving p'_i along the ray spanned by n'_i. As argued above, the statement is true for $t = 0$. It therefore suffices to show that $\phi(\partial P^\circ)$ never crosses the origin as $t \to 1$ because then the degree of $\phi(\partial P^\circ)$ around the origin stays nonzero, and hence, $0 \in \phi(\mathrm{int}(P^\circ))$ also at $t = 1$.

For sake of contradiction, assume that $\phi(\partial P^\circ)$ crosses the origin at t, that is, there is a proper face $f \subset P^\circ$ with

$$0 \in \phi(f) \overset{(*)}{\subseteq} \mathrm{conv}\{p'_i(t) \mid p_i \in f\}.$$

Let H be a supporting hyperplane for f in P'°. Moving H to the origin yield a hyperplane that has all $p'_i(0)$ with $p_i \in f$ strictly on one side. This stays true throughout the homotopy since all scaling factors are strictly positive. But then 0 cannot have been in the convex hull of these $p'_i(t)$.

References

[1] Y. Bi. Dihedral rigidity for convex polytopes by smooth approximation. *arXiv preprint arXiv:2608.06320*, 2026.

[2] J. Wang and Z. Xie. On gromov's flat corner domination conjecture and stoker's conjecture. *arXiv preprint arXiv:2203.09511*, 2022.

[3] M. Winter. Rigidity, tensegrity, and reconstruction of polytopes under metric constraints. *International Mathematics Research Notices*, 2024(9):7721–7747, 2024.

Max-Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Gernany

Email address: martin.winter@mis.mpg.de